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# LAPLACE TRANSFORM

$t > 0$

$$L\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt.$$

Properties:

$$L\{c\,F(t)\} = c\,L\{F(t)\}$$

$$L\{c_1 F_1(t) + c_2 F_2(t)\} = c_1 L\{F_1(t)\} + c_2 L\{F_2(t)\}$$

$$L(t) = \frac{1}{s}$$

$$L(t) = \frac{1}{s^2}$$

L(

If $t$ is greater than 0, and $F(t)$ any fn of $t$ then the definite integral of the form $\int_0^\infty e^{-st} F(t)\, dt$. If its exist and finite is called laplaces transform of $f(t)$ which is denoted by the symbol $L\{f(t)\}$.

$$\therefore L\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt$$

Properties:

$L\{cF(t)\} = c\,L\{F(t)\}$

$L\{c_1F_1(t) + c_2F_2(t)\} = c_1L\{F_1(t)\} + c_2L\{F_2(t)\}$

Some important formulaes

1. $L(1) = \frac{1}{s}$
2. $L(t) = \frac{1}{s^2}$
3. $L(t^n) = \frac{n!}{s^{n+1}}$ $= \frac{\Gamma_{n+1}}{s^{n+1}}$
4. $L(e^{at}) = \frac{1}{s-a}$
5. $L(e^{-at}) = \frac{1}{s+a}$
6. $L(\sin at) = \frac{a}{s^2+a^2}$
7. $L(\cos at) = \frac{s}{s^2+a^2}$
8. $L(\sinh at) = \frac{a}{s^2-a^2}$
9. $L(\cosh at) = \frac{s}{s^2-a^2}$

Note:

$\Gamma_n = \int_0^\infty e^{-t} t^{n-1}\, dt$

$\Gamma_n = \lfloor n-1$ if $n$ is an +ve integer $= (n-1)(n-2)\ldots$

$\Gamma_{\frac{1}{2}} = \sqrt{\pi}$

$\lfloor n = n!$

. Find laplace transform of the following functions.

$F(t) = \frac{t}{\tau}$ $\quad 0 \le t < \tau$

$= 1$ $\quad t \ge \tau$

$L\{F(t)\} = \int_0^\infty e^{-st} \cdot F(t)\, dt$

$= \int_0^\tau e^{-st} \cdot \frac{t}{\tau} \cdot dt + \int_\tau^\infty e^{-st} \cdot \frac{t}{\tau}\, dt$

$= \frac{1}{\tau}\left[\int_0^\tau e^{-st} \cdot t\, dt\right] + \frac{1}{-s}\, e^{-st}\Big\}_\tau^\infty$

$= \frac{1}{\tau}\left[t\, \frac{e^{-st}}{-s}\Big\}_0^\tau - \int_0^\tau \frac{e^{-st}}{-s} \cdot 1\, dt\right] + \frac{-1}{s}\left\{e^{-\infty} - e^{-s\tau}\right\}$

$= \frac{1}{\tau}\left[-\frac{1}{s}\left\{\tau e^{-s\tau} - 0\right\} + \frac{1}{s} \cdot \frac{-1}{s}\, e^{-st}\Big\}_0^\tau\right]$

$\frac{1}{s}\left\{0 - e^{-s\tau}\right\}$

$= \frac{1}{\tau}\left[\frac{-\tau}{s}\, e^{-s\tau} - \frac{1}{s^2}\left(e^{-s\tau} - e^0\right)\right]$

$+ \frac{1}{s}\, e^{-s\tau}$

$$= -\frac{1}{s}e^{-s\tau} - \frac{1}{s^2\tau}(e^{-s\tau}-1) + \frac{1}{s}e^{-s\tau}$$

$$= \frac{1}{s^2\tau}(1-e^{-s\tau})$$

$$\left( e^{-\infty} = \frac{1}{e^{\infty}} = \frac{1}{\infty} = 0 \right)$$

2. $F(t) = e^t \quad 0 < t < 1$

$\quad = 0 \quad t \geq 1$

$$L\,f(t) = \int_0^{\infty} e^{-st} F(t)\,dt$$

$$= \int_0^{1} e^{-st} \overset{(e^t)}{F(t)}\,dt + \int_1^{\infty} e^{-st} F(t)\,dt$$

$$= \int_0^{1} e^{-(s-1)t}\,dt + 0$$

$$= -\frac{1}{(s-1)} \left[ e^{-(s-1)t} \right]_0^1$$

$$= -\frac{1}{s-1}\left[ e^{-(s-1)} - e^0 \right]$$

$$= -\frac{1}{s-1}\left[ e^{-(s-1)} - 1 \right]$$

$$= \frac{1}{s-1}\left[ 1 - e^{-(s-1)} \right]$$

3. $F(t) = \sin 2t \quad 0 < t < \pi$

$\quad = 0 \quad t \geq \pi$

$$L\{F(t)\} = \int_0^{\infty} e^{-st} F(t)\,dt$$

$$= \int_0^{\pi} e^{-st}.\sin 2t.dt + \int_{\pi}^{\infty} e^{-st}\,0\,dt$$

$$= \int_0^{\pi} e^{-st}.\sin 2t\,dt$$

$$\text{Let } I = \left[ e^{-st}\frac{-1}{2}\cos 2t \right]_0^{\pi} - \int_0^{\pi} -\frac{1}{2}\cos 2t\,(-s e^{-st})\,dt.$$

$$= -\frac{1}{2}\left[ e^{-s\pi}\cos 2\pi - e^{0}\cos 0 \right] - \frac{s}{2}\int^{\pi} e^{-st}\cos 2t\,dt.$$

$$= -\frac{1}{2}\left[e^{-s\pi} - 1\right] - \frac{s}{2}\left[e^{-st}\frac{1}{2}\sin 2t\Big|_0^{\pi} - \int_0^{\pi}\frac{1}{2}\sin 2t\,(-se^{-st})\right]$$

$$= \frac{1}{2}\left(1 - e^{-s\pi}\right) - \frac{s}{2}\left[(0-0) - \frac{s}{2}\int_0^{\pi} e^{-st}\sin 2t\,dt\right]$$

$$= \frac{1}{2}\left[1 - e^{-s\pi}\right] - \frac{s^2}{4}\cdot I$$

$$= I + \frac{s^2}{4}I = \frac{1}{2}\left[1 - e^{-s\pi}\right]$$

$$I\left[1 + \frac{s^2}{4}\right] = \frac{1}{2}\left[1 - e^{-s\pi}\right]$$

$$I\left[\frac{4+s^2}{4}\right] = \frac{1}{2}\left[1 - e^{-s\pi}\right]$$

$$I = \frac{2}{s^2+4}\left[1 - e^{-s\pi}\right]$$

$$L\{F(t)\} = \frac{2}{s^2+4}\left(1 - e^{-s\pi}\right)$$

4. $F(t) = 1 \quad 0 < t < 2$

$\quad = t \quad t \geq 2$

$$L\{f(t)\} = \int_0^{\infty} e^{-st}\cdot F(t)\,dt$$

$$= \int_0^{2} e^{-st}\cdot 1\,dt + \int_2^{\infty} e^{-st}\cdot t\,dt$$

$$= \cancel{\left\{\frac{e^{-st}}{-s}\right\}_0^2} - \int_0^2 \left(\frac{e^{-st}}{-s}\right)dt$$

$$= \frac{1}{-s}\left(e^{-s2} - e^{0}\right) + \left(\left\{\frac{te^{-st}}{-s}\right\}_2^{\infty} - \int_2^{\infty}\frac{e^{-st}}{-s}\cdot 1\,dt\right)$$

$$= \frac{-1}{s}\left(e^{-2s} - 1\right) + \frac{2}{s}e^{-2s} - \frac{1}{s^2}\left(0 - e^{-2s}\right)$$

$$= -\frac{1}{s}e^{-2s} + \frac{1}{s} + \frac{2}{s}e^{-2s} + \frac{1}{s^2}e^{-2s}$$

$$= \frac{1}{s} + \frac{1}{s}e^{-2s} + \frac{1}{s^2}e^{-2s}$$

5. $F(t) = t \quad 0 < t < 4$

$\quad = 5 \text{ for } t \geq 4$

$$L\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt$$

$$= \int_0^4 e^{-st}\, t\, dt + \int_4^\infty e^{-st}\, 5\, dt$$

$$= \left(\left[\frac{t e^{-st}}{-s}\right]_0^4 - \int_0^4 \frac{e^{-st}}{-s}\, dt\right] + \frac{5}{-s}\left(e^{-s\infty} - e^{-5s}\right)$$

$$= -\frac{4}{s} e^{-4t} - \frac{1}{-s}\left[-\frac{1}{s} e^{-st}\right\}_0^4 - \frac{5}{s}\left(0 - e^{-4s}\right)$$

$$= -\frac{4}{s} e^{-4\cancel{t}} - \frac{1}{s^2}\left[e^{-4t} - e^{0}\right] + \frac{5}{s} e^{-4s}$$

$$= \frac{1}{s} e^{4s} - \frac{1}{s^2}\left(e^{4s} - 1\right)$$

6. $L\left(1 + 2t^3 - 4e^{3t} + 5e^{t}\right)$

$$= L(1) + 2L(t^3) - 4L(e^{3t}) + 5L(e^{t})$$

$$= \frac{1}{s} + 2\frac{3!}{s^{3+\cancel{1}}} - 4\frac{1}{s-3} + 5\frac{1}{s+1}$$

$$= \frac{1}{s} + \frac{2}{s^4} - \frac{4}{s-3} + \frac{5}{s+1}$$

7. $L\left[3\cosh 4t + 4\sin 3t\right]$

$$= 3L[\cosh 4t] + 4L[\sin 3t]$$

$$= 3\frac{s}{s^2-4^2} + 4\frac{3}{s^2+3^2}$$

$$= \frac{3s}{s^2-16} + \frac{12}{s^2+9}$$

8. $L\left(4\sinh 5t - 4\cos 4t\right)$

$$= 4L[\sin 5t] - 4L[\cos 4t]$$

$$= 4\frac{5}{s^2-5^2} - 4\times\frac{s}{s^2+5^2}$$

$$= \frac{20}{s^2-25} - \frac{\cancel{20}\,4s}{s^2+\cancel{25}\,16}$$

9. $L\left[at+bt\right]^3 = L\left[\cancel{a^3t^3 + 3a^2t^2bt + 3atb^2t^2 + b^3t^3}\right]$

$$= \cancel{a^3L\left[t^3 + 3a^2b + 3ab^2 + b^3\right.}$$

$= \cancel{a^3 L(t^3) + 3a^2 b L(t^3)}$

$- L\{(a+b)^3 t^3\}$

$= (a+b)^3 L(t^3)$

$= (a+b)^3 \frac{\lfloor 3}{s^{3+1}} = \frac{6(a+b)^3}{s^4}$

10. $L[a+bt]^3 = L[a^3 + 3a^2bt + 3b^2t^2a + b^3t^3]$

$= a^3 L(1) + 3a^2b\, L(t) + 3ab^2 L(t^2) + b^3 L(t^3)$

$= a^3 \cdot \frac{1}{s} + \frac{3a^2b}{s^2} + \frac{6ab^2}{s^3} + \frac{6b^3}{s^4}$

11. $L[3e^{-2t} - 2e^{+3t}] = 3L(e^{-2t}) - 2L(e^{3t})$

$= 3 \times \frac{1}{s+2} - 2\frac{1}{s-3}$

$= \frac{3}{s+2} - \frac{2}{s-3}$

12. $L(1+e^t)^2 = L[1 + 2e^t + e^{2t}]$

$= L(1) + 2L(e^t) + L(e^{2t})$

$= \frac{1}{s} + \frac{2}{s-1} + \frac{1}{s-2}$

13. $L[\cos(at+b)] = L\{\cos at \cdot \cos b - \sin at \sin b\}$

$= L(\cos at \cos b) - L(\sin at \sin b)$

$= \cos b\, L(\cos at) - \sin b\, L(\sin at)$

$= \cos b \times \frac{s}{s^2+a^2} - \sin b \frac{a}{s^2+a^2}$

$= \frac{1}{s^2+a^2}\{s\cos b - a\sin b\}$

NOTE:

$2\cos A \cos B : \cos(A+B) + \cos(A-B)$

$2\sin A \sin B : \cos[A-B] - \cos[A+B]$

$2\sin A \cos B : \sin(A+B) + \sin(A-B)$

$2\cos A \sin B : \sin(A+B) - \sin(A-B)$.

14. $L[\cos at \cos bt]$

$$\cos at \cos bt = \frac{1}{2} \times 2 \cos at \cos bt$$

$$= \frac{1}{2}\{\cos(at+bt) + \cos(at-bt)\}$$

$$= \frac{1}{2}\{\cos(a+b)t + \cos(a-b)t\}$$

$$L[\cos at \cos bt] = \frac{1}{2} L(\cos(a+b)t) + \frac{1}{2} L(\cos(a-b)t)$$

$$= \frac{1}{2} \times \frac{s}{s^2+(a+b)^2} + \frac{1}{2}\,\frac{s}{s^2+(a-b)^2}$$

$$= \frac{s}{2}\left[\frac{1}{s^2+(a+b)^2} + \frac{1}{s^2+(a-b)^2}\right]$$

15. $L(\sin at \sin bt)$

$$\sin at \sin bt = \frac{1}{2}\, 2 \sin at \sin bt$$

$$= \frac{1}{2}\{\cos(a-b)t - \cos(a+b)t\}$$

$$L(\sin at \sin bt) = \frac{1}{2} L\{\cos(a-b)t\} - \frac{1}{2} L(\cos(a+b)t)$$

$$= \frac{s}{2}\left[\frac{1}{s^2+(a-b)^2} - \frac{1}{s^2+(a+b)^2}\right].$$

16. $L(\cos at \sin bt)$

$$\cos at \sin bt = \frac{1}{2}\, 2 \cos at \sin bt$$

$$= \frac{1}{2}\{\sin(at+bt) - \sin(at-bt)\}$$

$$= \frac{1}{2}\{\sin(a+b)t - \sin(a-b)t\}$$

$$L(\cos at \sin bt) = \frac{1}{2} L(\sin(a+b)t) - \frac{1}{2} L(\sin(a-b)t)$$

$$= \frac{1}{2} \times \frac{a+b}{s^2+(a+b)^2} - \frac{1}{2}\,\frac{(a-b)}{s^2+(a-b)^2}$$

NOTE:

$$\cosh at = \frac{e^{at}+e^{-at}}{2}$$

$$\sinh at = \frac{e^{at}-e^{-at}}{2}.$$

17. $L(\cosh^2 at) = L\left(\frac{e^{at}+e^{-at}}{2}\right)^2$

$$= \frac{1}{4} L\left(e^{2at} + 2 e^{at} e^{-at} + e^{-2at}\right)$$

$$= \frac{1}{4} L\left(e^{2at} + 2 + e^{-2at}\right)$$

$$= \frac{1}{4}\left\{L\left(e^{2at}\right] + 2L(1] + L\left(e^{-2at}\right)\right]$$

$$= \frac{1}{4}\left[\frac{1}{s-2a} + \frac{2}{s} + \frac{1}{s+2a}\right]$$

18. $L\left(\sinh^2 at\right) = L\left[\frac{e^{at} - e^{-at}}{2}\right]^2$

$$= \frac{1}{4} L\left[e^{2at} - 2e^{at}e^{-at} + e^{-2at}\right]$$

$$= \frac{1}{4}\left[L\left(e^{2at}\right] - 2L(1) + L\left(e^{-2at}\right)\right]$$

$$= \frac{1}{4}\left[\frac{1}{s-2a} - \frac{2}{s} + \frac{1}{s+2a}\right]$$

19. $L\left[\cosh^3 at\right] = L\left[\frac{e^{at} + e^{-at}}{2}\right]^3$

$$= \frac{1}{8} L\left[e^{3at} + 3e^{2at} \cdot e^{-at} + 3e^{-2at} \cdot e^{at} + e^{-3at}\right]$$

$$= \frac{1}{8}\left\{L\left(e^{3at}\right) + 3L\left(e^{at}\right) + 3L\left(e^{-at}\right) + L\left(e^{-3at}\right)\right\}$$

$$= \frac{1}{8}\left[\frac{1}{s-3a} + \frac{3}{s-a} + \frac{3}{s+a} + \frac{1}{s+3a}\right]$$

20. $L\left(\sinh^3 at\right) = L\left[\frac{e^{at} - e^{-at}}{2}\right]^3$

$$= \frac{1}{8}\left[L\left(e^{3at}\right) - 3L\left(e^{at}\right) + 3L\left(e^{-at}\right) - L\left(e^{-3at}\right)\right]$$

$$= \frac{1}{8}\left[\frac{1}{s-3a} - \frac{3}{s-a} + \frac{3}{s+a} - \frac{1}{s+3a}\right]$$

NOTE:

$\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$

$\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$

$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

21. $L(\sin^2 t) = L\left[\frac{1}{2} - \frac{1}{2}\cos 2t\right]$

$$= \frac{1}{2}L(1] - \frac{1}{2}L(\cos 2t)$$

$$= \frac{1}{2s} - \frac{1}{2}\frac{s}{s^2+2^2}$$

$$= \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)$$

22. $L(\cos^2 t) = L\left[\frac{1}{2} + \frac{1}{2}\cos 2t\right]$

$= \frac{1}{2} \times \frac{1}{s} + \frac{1}{2}\frac{s}{s^2+4}$

$= \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+4}\right]$

23. $L(\sin^3 at)$

$\sin 3at = 3\sin at - 4\sin^3 at$

$\sin^3 at = \frac{3}{4}\sin at - \frac{1}{4}\sin 3at$

$L(\sin^3 at) = \frac{3}{4}L[\sin at] - \frac{1}{4}L(\sin 3at)$

$= \frac{3}{4}\cdot\frac{a}{s^2+a^2} - \frac{1}{4}\frac{3a}{s^2+(3a)^2}$

$= \frac{3}{4}a\left[\frac{1}{s^2+a^2} - \frac{a}{s^2+9a^2}\right]$

24. $L(\cos^3 at)$

$\cos^3 at = \frac{3}{4}\cos at + \frac{1}{4}\cos 3at$

$L[\cos^3 at] = \frac{3}{4}L[\cos at] + \frac{1}{4}L[\cos 3at]$

$= \frac{3}{4}\cdot\frac{s}{s^2+a^2} + \frac{1}{4}\frac{s}{s^2+(3a)^2}$

$= \frac{s}{4}\left[\frac{3}{s^2+a^2} + \frac{1}{s^2+9a^2}\right]$

25. $L[\cos^3 2t]$ $\qquad [2t=\theta]$

$\cos 6t = \cos 3.2t$

$= \cos 3\theta$

$= 4\cos^3\theta - 3\cos\theta$

$3\cos\theta + \cos 6t = 4\cos^3\theta$

$3\cos 2t + \cos 6t = 4\cos^3 2t$

$\cos^3 2t = 3/4\cos 2t + 1/4\cos 6t$

$L[\cos^3 2t] = 3/4\, L[\cos 2t] + 1/4\, L(\cos 6t)$

$= 3/4 \times \frac{s}{s^2+2^2} + 1/4 \times \frac{6}{s^2+6^2}$

26. $\sin 6t = \sin 3.2t$ $\qquad \theta = 2t$

$= \sin 3\theta$

$= 3\sin\theta - 4\sin^3\theta$

$4\sin^3\theta = 3\sin\theta - \sin 6t$

$4\sin^3 2t = 3\sin 2t - \sin 6t$

$\sin^3 2t = 3/4\sin 2t - 1/4\sin 6t$

$L[\sin^3 2t] = 3/4\, L[\sin 2t] - 1/4\, L[\sin 6t]$

$= 3/4 \ L[\sin 2t] - 1/4 \ L[\sin 6t]$

$= 3/4 \times \frac{2}{s^2+2^2} - 1/4 \times \frac{6}{s^2+6^2}$

$= \frac{6}{4}\left[\frac{1}{s^2+2^2} - \frac{1}{s^2+6^2}\right]$

$= \frac{3}{2}\left(\frac{32}{(s^2+2^2)(s^2+6^2)}\right)$

$= \frac{48}{(s^2+2^2)(s^2+6^2)}$

27. $L[\cos^2 2t]$

$\cos 4t = \cos 2.2t$ $\qquad 2t = \theta$

$= \cos 2\theta$

$= 2\cos^2\theta - 1$

$1 + \cos 4t = 2\cos^2 2t$

$\cos^2 2t = 1/2 + 1/2 \cos 4t$

$L[\cos^2 2t] = L(1/2) + 1/2 \ L[\cos 4t]$

$= 1/2 \ L(1) + 1/2 \ L[\cos 4t]$

$= \frac{1}{2s} + \frac{1}{2}\frac{s}{s^2+4^2}$

28. $\sin^2 2t = 1/2 - 1/2 \cos 4t$

$L(\sin^2 2t) = 1/2 \ L(1) - 1/2 \ L(\cos 4t)$

$= \frac{1}{2s} - \frac{1}{2}\frac{s}{s^2+4^2}$

$= 1/2\left(\frac{1}{s} - \frac{s}{s^2+4^2}\right)$

## DIFFERENT KINDS OF FINDING LAPLACE'S TRANSFORMATION

TYPE I - BY SHIFTING RULE [FIRST SHIFTING PROPERTY]

If $L\{F(t)\} = F(s)$, then

$L\{e^{at}F(t)\} = F[s-a]$

Proof: $L\{F(t)\} = \int_0^\infty e^{-st}.F(t).dt = F(s)$

$= \int_0^\infty e^{-st}.e^{at}F(t).dt$

$= \int_0^\infty e^{-[s-a]t}.F(t).dt$

$= f(s-a)$

$\therefore L\{e^{-at} F(t)\} = F(s+a)$

1) $L[e^{at}] = L[e^{at} \cdot 1]$

$= \dfrac{1}{s-a}$

2) $L(e^{-at}) = \dfrac{1}{s+a}$

3) $L(e^{at} t^n) = \dfrac{\lfloor n}{(s-a)^{n+1}} = \dfrac{\Gamma(n+1)}{(s-a)^{n+1}}$

4) $L[e^{-at} t^n] = \dfrac{\lfloor n}{(s+a)^{n+1}} = \dfrac{\Gamma(n+1)}{(s+a)^{n+1}}$

5) $L[e^{at} \sin bt] = \dfrac{b}{(s-a)^2+b^2}$

6. $L[e^{-at} \sinh bt] = \dfrac{b}{(s+a)^2+b^2}$

7. $L[e^{at} \sinh bt] = \dfrac{b}{(s-a)^2-b^2}$

8. $L[e^{at} \cosh bt] = \dfrac{s-a}{(s-a)^2-b^2}$

29 $L[e^{2t} t^5] = \dfrac{\lfloor 5}{(s-2)^{5+1}} = \dfrac{120}{(s-2)^6}$

30. $L\{e^{-t}(3\cosh 2t - 2\cosh 3t)\}$

$= 3L(e^{-t}\cosh 2t) - 2L(e^{-t}\cosh 3t)$

$= 3 \times \dfrac{s+1}{(s+1)^2-4} - 2 \times \dfrac{s+1}{(s+1)^2-3^2}$

$= s+1\left[\dfrac{3}{(s+1)^2-4} - \dfrac{2}{(s+1)^2-3^2}\right]$

31. $L[e^{3t}\sin^2 t] = L[e^{3t}(\tfrac{1}{2} - \tfrac{1}{2}\cos 2t)]$

$= \tfrac{1}{2} L(e^{3t}) - \tfrac{1}{2} L(e^{3t}\cos 2t)$

$= \dfrac{1}{2}\cdot\dfrac{1}{s-3} - \dfrac{1}{2}\cdot\dfrac{s-3}{(s-3)^2+4}$

32. $= \dfrac{-2}{(s-3)((s-3)^2-4)}$

32. $L(e^{-t}(3\cosh 2t - 2\cosh 3t)\}$

$= 3L(e^{-t}\cosh 2t) - 2L(e^{-t}\cosh 3t)$

$$= 3 \times \frac{s+1}{(s+1)^2 - 2^2} - \frac{2 \times (s+1)}{(s+1)^2 - 3^2}$$

$$= s+1 \left( \frac{3}{(s+1)^2 2^2} - \frac{2}{(s+1)^2 - 3^2} \right)$$

33. $L\left(e^{-4t} \cdot \cos t \sin 2t\right)$

$$\cos t \sin 2t = 1/2 \times 2 \sin 2t \cos t$$

$$= \frac{1}{2}\left(\sin(2t+t) + \sin(2t-t)\right)$$

$$= 1/2 \sin 3t + 1/2 \sin t$$

$$L\left[e^{-4t} \cos t \sin 2t\right] = 1/2\ L\left(e^{-4t} \sin 3t\right) +$$

$$1/2\ L\left(e^{-4t} \sin t\right)$$

$$= 1/2 \times \frac{3}{(s+4)^2 + 3^2} + 1/2 \frac{1}{(s+4)^2 + 1^2}$$

34. $L\left[e^{at} \cdot \sin t \cdot \sin 2t \cdot \sin 3t\right]$

$$\sin t \sin 2t \sin 3t = \frac{1}{2} \times \cdot 2 \sin 3t \sin 2t \sin t$$

$$= \frac{1}{2}\left(\cos(3t-2t) - \cos(3t+2t)\right) \sin t$$

$$= \frac{1}{2} \cos t \sin t - 1/2 \cos 5t \sin t$$

$$= \frac{1}{4} 2 \cos t \sin t - \frac{1}{4} 2 \cos 5t \sin t$$

$$= \frac{1}{4} \sin 2t - \frac{1}{4}\left(\sin(5t+t) - \sin(5t-t)\right)$$

$$= \frac{1}{4} \sin 2t - \frac{1}{4} \sin 6t + 1/4 \sin 4t$$

$$L\left(e^{at} \sin t \sin 2t \sin 3t\right) =$$

$$= 1/4\ L\left(e^{at} \sin 2t\right) - 1/4\ L\left(e^{at} \sin 6t\right) + 1/4\ L\left(e^{at} \sin 4t\right)$$

$$= \frac{1}{4} \times \frac{2}{(s-a)^2 + 2^2} - 1/4 \times \frac{6}{(s-a)^2 + 6^2} + 1/4 \frac{4}{(s-a)^2 + 4^2}$$

$$= \frac{1}{2} \times \frac{1}{(s-a)^2 + 2^2} - \frac{3}{2} \times \frac{1}{(s-a)^2 + 36} + \frac{1}{(s-a)^2 + 16}$$

35. $L(\sqrt{t}) = L(t^{1/2}) = \frac{\Gamma(1/2 + 1)}{s^{1/2+1}}$

$$= \frac{1/2\ \Gamma 1/2}{s^{3/2}}$$

$$\Gamma n = \int_0^\infty e^{-t} t^{n-1} dt$$

$$= (n-1)\Gamma(n-1)$$

$$= \lfloor n-1$$

$$\Gamma 1/2 = \sqrt{\pi}$$

$$= \frac{\frac{1}{2}\sqrt{\pi}}{s^{3/2}}$$

$$L\left[\frac{1}{\sqrt{t}}\right] = L\left[\frac{1}{t^{1/2}}\right] = L\left(t^{-1/2}\right)$$

$$= \frac{\Gamma\left(-\frac{1}{2}+1\right)}{s^{-1/2+1}}$$

$$= \frac{\Gamma\left(\frac{1}{2}\right)}{s^{1/2}}$$

$$= \frac{\sqrt{\pi}}{\sqrt{s}}$$

$$\Rightarrow L\left(t^{3/2}\right) = \frac{\Gamma\left(\frac{3}{2}+1\right)}{s^{3/2+1}}$$

$$= \frac{\frac{3}{2}\Gamma\left(\frac{3}{2}\right)}{s^{5/2}}$$

$$= \frac{3}{2} \times \frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{3}{4}\frac{\sqrt{\pi}}{s^{5/2}}$$

$$\Rightarrow L\left[\frac{1}{t^{3/2}}\right] = L\left[t^{-3/2}\right]$$

$$= \frac{\Gamma\left(-\frac{3}{2}+1\right)}{s^{-3/2+1}}$$

$$= \frac{\Gamma\left(-\frac{1}{2}\right)}{s^{1/2}} = \frac{-2\sqrt{\pi}}{s^{1/2}}$$

$$\Rightarrow L\left(\sqrt{t}+\frac{1}{\sqrt{t}}\right)^3 = L\left(t^{1/2}+t^{-1/2}\right)^3$$

$$= L\left[t^{3/2} + 3t\,t^{-1/2} + 3t^{1/2}t^{-1} + t^{-3/2}\right]$$

$$= L\{t^{3/2}\} + 3t^{1/2} + 3t^{-1/2} + t^{-3/2}]$$

$$= \frac{3}{4}\frac{\sqrt{\pi}}{s^{5/2}} + 3\times\frac{1}{2}\frac{\sqrt{\pi}}{s^{3/2}} + 3\times\frac{\sqrt{\pi}}{\sqrt{s}} + \frac{-2\sqrt{\pi}}{s^{-1/2}}$$

$$= \frac{\sqrt{\pi}}{\sqrt{s}}\left(\frac{1}{4\times s^2} + \frac{3}{2s}\right] - 2\sqrt{\pi}\,s$$

Note:

$$\Gamma\left(\tfrac{1}{2}\right) = \Gamma\left(-\tfrac{1}{2}+1\right)$$

$$\sqrt{\pi} = -\tfrac{1}{2}\,\Gamma\left(-\tfrac{1}{2}\right)$$

$-2\sqrt{\pi} = \Gamma -1/2$

TYPE II - MULTIPLICATION BY t

$L[F(t)] = F(s)$

$L[t^n \cdot f(t)] = -1^n \frac{d^n f(s)}{d s^n}$

$= (-1)^n \frac{d^n}{d s^n} [L(F(t))]$

Proof: $L[F(t)] = \int_0^\infty e^{-st} F(t)\,dt$

$f(s) = \int_0^\infty e^{-st} f(t)\,dt$

$\frac{d}{ds}(f(s)) = \frac{d}{ds} \int_0^\infty e^{-st} f(t)\,dt$

$= \int_0^\infty \frac{d}{ds} \cdot (e^{-st}) \cdot F(t)\,dt$

$= \int_0^\infty (-t) e^{-st} \cdot F(t)\,dt$

$= -\int_0^\infty e^{-st} (t F(t))\,dt$

$-\frac{d}{ds} F(s) = \int_0^\infty e^{-st} [t F(t)]\,dt$

$= L(t F(t))$

$(-1)^2 \frac{d^2}{ds^2} \{f(s)\} = \frac{d}{ds} \left\{\frac{d}{ds} f(s)\right\}$

$= \frac{d}{ds} - \int_0^\infty [e^{-st} t F(t)\,dt]$

$= -\frac{d}{ds} \int_0^\infty e^{-st} t \cdot F(t) \cdot dt$

$= -\int_0^\infty -t\, e^{-st} t \cdot F(t)\,dt$

$= \int_0^\infty e^{-st} t^2 F(t)\,dt$

$= L(t^2 F(t))$

$L[t^2 F(t)] = (-1)^2 \frac{d^2}{ds^2} L[f(t)]$

$L\{t^3 F(t)\} = (-1)^3 \frac{d^3}{ds^3} L(F(t))$

$$\therefore L\{t^n F(t)\} = (-1)^n \frac{d^n}{ds^n} L[F(t)]$$

$$L[e^{-t} \sin 4t + t\cos 2t]$$

$$= L[e^{-t} \sin 4t] + L[t\cos 2t]$$

$$= \frac{4}{(s+1)^2+4^2} + (-1)\frac{d}{ds} L[\cos 2t]$$

$$= \frac{4}{(s+1)^2+16} - \frac{d}{ds}\frac{s}{s^2+2^2}$$

$$= \frac{4}{(s+1)^2+16} - \left(\frac{s^2+4 \cdot 1 - s(2s)}{(s^2+4)^2}\right)$$

$$= \frac{4}{(s+1)^2+16} - \frac{4-s^2}{[s^2+4]^2}$$

$$= \frac{4}{(s+1)^2+16} + \frac{(s^2-4)}{(s^2+4)^2}$$

$$L[t^2 \times \sin at] = (-1)^2 \frac{d^2}{ds^2}\frac{a}{s^2+a^2}$$

$$= a\frac{d}{ds}\frac{d}{ds}\frac{1}{s^2+a^2}$$

$$= a\frac{d}{ds}\frac{(-1)}{(s^2+a^2)^2} \cdot 2s$$

$$= -2a\frac{d}{ds}\frac{s}{(s^2+a^2)^2}$$

$$= -2a\frac{(s^2+a^2)^2 \cdot 1 - s \cdot 2(s^2+a^2)2s]}{[s^2+a^2]^4}$$

$$= -2a\frac{s^2+a^2\,[s^2+a^2-4s^2]}{(s^2+a^2)^4}$$

$$= \frac{-2a(a^2-3s^2]}{[s^2+a^2]^3}$$

$$L(t \cdot e^{-t} \sin 4t) = -1\frac{d}{ds} L(e^{-t}\sin 4t)$$

$$= -\frac{d}{ds}\frac{4}{(s+1)^2+4^2}$$

$$= -4\frac{d}{ds}\frac{1}{(s+1)^2+4^2}$$

$$= -4 \times \frac{-1}{\{(s+1)^2+16\}^2} \cdot 2(s+1)$$

$$= \frac{8(s+1)}{[(s+1)^2+16]^2}$$

$L\left(t \cdot e^{-3t} \cdot \cos 2t\right)$

$$= (-1)\frac{d}{ds} L\left[e^{-3t}\cos 2t\right]$$

$$= -\frac{d}{ds}\frac{s+3}{(s+3)^2+2^2}$$

$$= -\frac{(s+3)^2+2^2 - (s+3)[2(s+3)]}{[(s+3)^2+4]^2}$$

$$= -\frac{[(s+3)^2+4 - 2(s+3)^2]}{[(s+3)^2+4]^2}$$

$$= \frac{-4+[s+3]^2}{[(s+3)^2+4]^2}$$

$$= \frac{[s+3]^2-4}{[(s+3)^2+4]^2}$$

$L[t^2 \cos 2t]$

$$= [-1]^2 \frac{d}{ds} L[\cos 2t]$$

$$= \frac{d}{ds}\frac{d}{ds}\cos 2t$$

$$= \frac{d}{ds}\frac{d}{ds}\frac{s}{s^2+4}$$

$$= \frac{d}{ds}\left[\frac{s^2+4 \cdot - s \cdot 2s}{(s^2+4)^2}\right]$$

$$= \frac{d}{ds}\left(\frac{4-s^2}{(s^2+4)^2}\right)$$

$$= \frac{(s^2+4)^2 \cdot -2s - (4-s^2)\,2(s^2+4) \cdot 2s}{(s^2+4)^4}$$

$$= \frac{[s^2+4]\left[(s^2+4) \cdot -2s - (4-s^2) \cdot 4s\right]}{(s^2+4)^4}$$

$$= \frac{-2s^3 - 8s - 16s + 4s^3}{[s^2+4]^3}$$

$$= \frac{2s^3-24s}{(s^2+4)^3}$$

* $L[\cosh at \cdot \sin bt] = L\left[\frac{e^{at}+e^{-at}}{2}\right] \cdot \sin bt$

$= \frac{1}{2} L\left[e^{at} \sin bt\right] + \frac{1}{2} L\left(e^{-at} \sin bt\right)$

$= \frac{1}{2} \times \frac{b}{(s-a)^2+b^2} + \frac{1}{2} \frac{b}{(s+a)^2+b^2}$

* $L\left[\sinh at \cdot \cos bt\right] = L\left[\frac{e^{at}-e^{-at}}{2}\right] \cdot \cos bt$

$= \frac{1}{2} L\left[e^{at} \cos bt\right] - \frac{1}{2} L\left[e^{-at} \cos bt\right]$

$= \frac{1}{2} \times \frac{s-a}{(s-a)^2+b^2} - \frac{1}{2} \frac{s+a}{(s+a)^2+b^2}$

TYPE III - DIVISION BY $t$

If $L[f(t)] = F(s)$

the $L\left[\frac{f(t)}{t}\right] = \int_s^\infty f(s)\, ds$

$= \int_s^\infty L(F(t))\, ds$

Proof: $L(F(t)) = \int_0^\infty e^{-st} f(t)\, dt$

$\int_s^\infty L[f(t)]\, ds = \int_s^\infty \left[\int_0^\infty e^{-st} F(t)\, dt\right] ds$

$= \int_0^\infty \left[\int_s^\infty e^{-st}\, ds\right] F(t)\, dt$

$= \int_0^\infty \left\{-\frac{1}{t} e^{-st}\right\}_s^\infty \cdot F(t)\, dt$

$= \int_0^\infty -\frac{1}{2}\left\{e^{-\infty} - e^{-st}\right\} F(t)\, dt$

$= \int_0^\infty -\frac{1}{2}\left(0 - e^{st}\right) F(t)\, dt$

$= \int_0^\infty e^{-st} \frac{F(t)}{t}\, dt$

$= L\left(\frac{F(t)}{t}\right)$

$L\left(\frac{F(t)}{t}\right) = \int_s^\infty L[F(t)]\, ds.$

$\rightarrow L\left[\frac{\sin at}{t}\right] = \int_s^\infty L(\sin at)\, ds$

$$= \int_s^\infty \frac{a}{s^2+a^2}\, ds$$

$$= a \times \int_s^\infty \frac{1}{s^2+a^2}\, ds$$

$$= a \times \frac{1}{a} \left[\tan^{-1}\frac{s}{a}\right]_s^\infty$$

$$= \tan^{-1}\infty - \tan^{-1}\frac{s}{a}$$

$$= \frac{\pi}{2} - \tan^{-1}\frac{s}{a}$$

$$= \cot^{-1}\frac{s}{a}$$

* $L\left[\frac{\cos at - \cos bt}{t}\right]$

$$= \int_s^\infty \left[L(\cos at) - L(\cos bt)\right] ds$$

$$= \int_s^\infty \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right) ds$$

$$= \frac{1}{2}\int_s^\infty \left(\frac{2s}{s^2+a^2} - \frac{2s}{s^2+b^2}\right) ds$$

$$= \frac{1}{2}\left[\log(s^2+a^2) - \log(s^2+b^2)\right]$$

$$= \frac{1}{2}\left[\log\left|\frac{s^2+a^2}{s^2+b^2}\right|\right]_s^\infty$$

$$= \frac{1}{2}\log\left|\frac{s^2\left(1+\frac{a^2}{s^2}\right)}{\cancel{s}^2\left(1+\frac{b^2}{s^2}\right)}\right|_s^\infty$$

$$= \frac{1}{2}\left[\log\left|\frac{1+\frac{a^2}{\infty}}{1+b^2/\infty}\right| - \log\left|\frac{1+\frac{a^2}{s^2}}{1+\frac{b^2}{s^2}}\right|\right]$$

$$= \frac{1}{2}\left[\log 1 - \log\left|\frac{s^2+a^2}{s^2+b^2}\right|\right]$$

$$= \frac{1}{2}\left|0 + \log\frac{s^2+b^2}{s^2+a^2}\right|$$

$$= \log\sqrt{\frac{s^2+b^2}{s^2+a^2}} \;//$$

* $L\left[\frac{e^{-at} - e^{-bt}}{t}\right] = \int_s^\infty \left(L(e^{-at}) - L(e^{-bt})\right) ds.$

$$= \int_s^\infty \left[\frac{1}{s+a} - \frac{1}{s+b}\right] ds$$

$$= \left[\log\left|\frac{s+a}{s+b}\right|\right]_s^\infty$$

$$= \log\left(\frac{1+a/s}{1+b/s}\right]_s^\infty = \log\frac{1+a/\infty}{1+b/\infty} - \log\frac{1+a/s}{1+b/s}$$

$$= -\log\frac{s+a}{s+b}$$

$$= \log\frac{s+b}{s+a}$$

$$* \ L\left[\frac{1-e^{at}}{t}\right] = \int_s^\infty \{L(1) - L(e^{at})\}\, ds$$

$$= \int_s^\infty \left[\frac{1}{s} - \frac{1}{s-a}\right] \cdot ds$$

$$= \left[\log\frac{s}{s-a}\right]_s^\infty$$

$$= \left[\log\frac{1}{1-a/s}\right]_s^\infty$$

$$= \log 1 - \log\frac{s}{s-a}$$

$$\log\frac{s-a}{s}$$

$$* \ L\left[\frac{1-\cos at}{t}\right] = \int_s^\infty [L(1) - L(\cos at)]\, ds$$

$$= \int_s^\infty \left[\frac{1}{s} - \frac{s}{s^2+a^2}\right] \cdot ds$$

$$= \frac{1}{2}\int_s^\infty \frac{2}{s} - \frac{2s}{s^2+a^2} \cdot ds$$

$$= \frac{1}{2}\left[2\log s - \log s^2+a^2\right]_s^\infty$$

$$= \frac{1}{2}\left(\log\frac{s^2}{s^2+a^2}\right)_s^\infty$$

$$= \frac{1}{2}\left[\log\frac{1}{1+a^2/s^2}\right]_s^\infty$$

$$= \frac{1}{2}\left[\log 1 - \log\frac{s^2}{s^2+a^2}\right]$$

$$= \frac{1}{2}\log\frac{s^2+a^2}{s^2}$$

$$= \log\sqrt{\frac{s^2+a^2}{s^2}}$$

* $L\left[\frac{\sin^2 t}{t}\right] = \int_s^\infty L(\sin^2 t)\, ds$

$= \int_s^\infty L\left(\frac{1}{2} - \frac{1}{2}\cos 2t\right) . ds.$

$= \int_s^\infty \left[\frac{1}{2s} - \frac{1}{2}\left[\frac{s}{s^2+4}\right]\right] ds$

$= \frac{1}{4}\int_s^\infty \left[\frac{2}{s} - \frac{2s}{s^2+4}\right] ds$

$= \frac{1}{4} . \left|\log s^2 - \log(s^2+4)\right|_s^\infty$

$= \frac{1}{4}. \left[\log \frac{s^2}{s^2+4}\right]_s^\infty$

$= \frac{1}{4}\left[\log \frac{1}{1+4/s^2}\right]_s^\infty$

$= \frac{1}{4}\left[\log \frac{1}{1} - \log \frac{s^2}{s^2+4}\right]$

$= \frac{1}{4}. \log \left|\frac{s^2+4}{s^2}\right|$

$=$

* LAPLACE TRANSFORM OF DEFINITE INTEGRAL

If $L(F(t)) = f(s)$

then $L\left[\int_0^t F(t)\, dt\right] = 1/s \; L[f(s)]$

$= 1/s\, L[f(t)]$

Let $g(t) = \int_0^t F(t) dt$

$g(0) = \int_0^0 F(t) dt$

$= 0$

$g'(t) = F(t)$

$L[g(t)] = \int_0^\infty e^{-st} g(t) dt$

$= \left. g(t). \frac{-1}{s} e^{-st}\right\}_0^\infty - \int_0^\infty \frac{1}{-s} e^{-st} g'(t)\, dt$

$= \frac{-1}{s}[0-0] + 1/s \int_0^\infty e^{-st} F(t) dt$

$\therefore L\left[\int_0^t F(t). dt\right] = \frac{1}{s} L F(t)$

$* \; L\left[\int_0^t e^{-t}\cos t\,dt\right] = \frac{1}{s} L\left(e^{-t}\cos t\right)$

$= \frac{1}{s}\left(\frac{s+1}{(s+1)^2+1}\right)$

$* \; L\left[\int_0^t \left(\frac{\sin t}{t}\right).dt\right] = \frac{1}{s} L\left[\frac{\sin t}{t}\right]$

$= \frac{1}{s}\int_s^\infty L(\sin t)\,ds$

$= \frac{1}{s}\int_s^\infty \frac{1}{s^2+1}.ds$

$= \frac{1}{s}\left[\tan^{-1}(s)\right]_s^\infty$

$= \frac{1}{s}\left|\tan^{-1}\infty - \tan^{-1}(s)\right|$

$= \frac{1}{s}\left|\pi/2 - \tan^{-1}s\right|$

$= \frac{1}{s}\cot^{-1}s$

$* \; L\int_0^t e^t\frac{\sin t}{t}.dt = \frac{1}{s} L\, e^t\frac{\sin t}{t}$

$= \frac{1}{s}\int_s^\infty L\, e^t\sin t$

$= \frac{1}{s}\int_s^\infty \frac{1}{(s-1)^2+1^2}\,ds.$

$= \frac{1}{s}\left[\tan^{-1}s-1\right]_s^\infty$

$= \frac{1}{s}\left[\tan^{-1}\infty - \tan^{-1}s-1\right]$

$= \frac{1}{s}\left(\frac{\pi}{2} - \tan^{-1}(s-1)\right)$

$= \frac{1}{s}\cot^{-1}(s-1)$

$* \; L\left[\int_0^t t^2\sin at.dt\right] = \frac{1}{s}L(t^2\sin at)$

$= \frac{1}{s}(-1)^2\frac{d^2}{ds^2}L(\sin at)$

$= \frac{1}{s}\frac{d^2}{ds^2}\frac{a}{s^2+a^2}$

$= \frac{a}{s}\frac{d}{ds}\frac{d}{ds}\frac{a}{s^2+a^2}$

$= \frac{a}{s}\frac{d}{ds}\frac{s^2+a^2-2s}{(s^2+a^2)^2}$

$= \frac{a}{s}\frac{d}{ds}\frac{(1)}{(s^2+a^2)^2}.2s.$

$$= \frac{-2a}{s} \frac{d}{ds} \frac{s}{(s^2+a^2)^2}$$

$$= -2a\left[\frac{(s^2+a^2)^2 - s\, 2(s^2+a^2).\, 2s}{(s^2+a^2)^4}\right]$$

$$= \frac{-2a}{s}\left[\frac{a^2-3s^2}{(s^2+a^2)^3}\right] = \frac{2a}{s}\frac{(3s^2-a^2)}{(s^2+a^2)^3}$$

* $L\left[\int_0^t (t\, e^{-t}\sin 4t)\, dt\right] = \frac{1}{s} L\left(t\, e^{-t}\sin 4t\right)$

$$= \frac{1}{s}(-1)\frac{d}{ds} L\left(e^{-t}\sin 4t\right)$$

$$= \frac{-1}{s}\frac{d}{ds}\frac{4}{(s+1)^2+4^2}$$

$$= \frac{-4}{s}\frac{d}{ds}\frac{1}{(s+1)^2+4^2}$$

$$= \frac{-4}{s} \times \frac{-1}{((s+1)^2+4^2)^2} \cdot 2(s+1)$$

$$= \frac{8(s+1)}{((s+1)^2+16)^2}$$

* $L\int_0^t \left[\frac{\cos at - \cos bt}{t}\right] dt$

$$= \frac{1}{s} L\left[\frac{\cos at - \cos bt}{t}\right]$$

$$= \frac{1}{s}\int_s^\infty L(\cos at - \cos bt)\, ds$$

$$= \frac{1}{s}\int_s^\infty \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\, ds$$

$$= \frac{1}{2s}\int_s^\infty \frac{2s}{s^2+a^2} - \frac{2s}{s^2+b^2}$$

$$= \frac{1}{2s}\left(\log \frac{s^2+a^2}{s^2+b^2}\right)_s^\infty$$

$$= \frac{1}{2s}\left(\log\left(\frac{1+a^2/s^2}{1+b^2/s^2}\right)\right)_s^\infty$$

$$= \frac{1}{2s}\left[\log 1 - \log\frac{s^2+a^2}{s^2+b^2}\right]$$

$$= \frac{1}{2s}\log\frac{s^2+b^2}{s^2+a^2}$$

$$= \frac{1}{s}\log\sqrt{\frac{s^2+b^2}{s^2+a^2}}$$

$$L\int_0^t \sin t \sin 2t \sin 3t \, dt$$

$$= \frac{1}{s} L(\sin t \sin 2t \sin 3t)$$

$$= \frac{1}{s}\left(\frac{1}{4}\left[\frac{2}{s^2+4} - \frac{6}{s^2+36} + \frac{4}{s^2+16}\right]\right)$$

## Laplace Transform for the Derivatives

$$L[f'(t)] = s\,L[F(t)] - F(0)$$

Proof. Also derive laplace transform for $n^{th}$ derivative.

Proof: $L[F'(t)] = \int_0^\infty e^{-st} F'(t) \, dt$

$$= e^{-st} F(t)\Big\}_0^\infty - \int_0^\infty F(t).(-s)e^{-st} dt$$

$$= 0 - e^{0} F(0) + s\int_0^\infty e^{-st} F(t) dt$$

$$= sL[F(t)] - F(0)$$

$$L[F''(t)] = sL[F'(t)] - F'(0)$$

$$= s[sL[F(t)] - F(0)] - F'(0)$$

$$= s^2 L[F(t)] - s(F(0)) - F'(0)$$

$$L[F'''(t)] = sL(F''(t)) - F''(0)$$

$$= s[s^2 L(F(t)) - sF(0) - F'(0)] - F''(0)$$

$$= s^3 Lf(t) - s^2F(0) - sF'(0) - F''(0)$$

$$L[F^n(t)] = s^n Lf(t) - s^{n-1}F(0) - sf^{n-2}(0) - f'(0)$$

## Periodic Functions:

A function $F(t)$ is said to be periodic and is of period $T$ [$T$ is a constant]. If $F(t+T)$ equals $F(t)$

eg: $\sin x$, $\cos x$ of periodic functions of period $2\pi$

If $f(x) = \sin x$  $f(x+2\pi) = \sin[x+2\pi] = \sin x$

$f(x+4\pi) = \sin[x+4\pi] = \sin x$.

If $f(x) = \cos x$.

$f(x+2\pi) = \cos(2\pi+x) = \cos x$

. Find the Laplace transform of periodic function $F(t)$ with period $T$

$$\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} . F(t) . dt$$

$$= \int_0^T e^{-st} F(t)\, dt + \int_T^{2T} e^{-st} F(t)\, dt + \int_{2T}^{3T} e^{-st} F(t)\, dt$$

Take $\int_T^{2T} e^{-st} F(t)\, dt$

Let $t = u + T$ $\quad dt = du + 0$

when $t = T$,

$T = u + T$

$0 = u$

When $t = 2T$

$2T = u + T$

$T = u.$

$$\int_T^{2T} e^{-st} F(t)\, dt = \int_0^T e^{-s(u+T)} . F(u+T)\, du$$

$$= \int_0^T e^{-su} . e^{-sT} F(u) . du$$

$$= e^{-sT} \int e^{-su} . F(u)\, du$$

$$= e^{-sT} \int_0^T e^{-st} F(t) . dt$$

$$\int_{2T}^{3T} e^{-st} F(t)\, dt = e^{-2sT} \int_0^T e^{-st} F(t)\, dt$$

$$\int_{3T}^{4T} e^{-st} F(t)\, dt = e^{-3sT} \int_0^T e^{-st} F(t)\, dt$$

$$\mathcal{L}\{F(t)\} = \int_0^T e^{-st} F(t)\, dt + e^{-sT} \int_0^T e^{-st} F(t)\, dt + e^{-2sT} \int_0^T e^{-st} F(t) . dt \dots$$

$$= \int_0^T e^{-st} F(t) . dt \left[ 1 + e^{-st} + e^{-2st} \dots \right]$$

$$= \int_0^T e^{-st} F(t)\, dt \times \frac{1}{1 - e^{-st}}$$

. Find the laplaces transform of periodic function $F(t)$ with period $T$

$$\mathcal{L}\{F(t)\} = \int_0^{\infty} e^{-st} . F(t) . dt$$

$$= \int_0^{T} e^{-st} F(t)\, dt + \int_T^{2T} e^{-st} F(t) . dt + \int_{2T}^{3T} e^{-st} F(t) \ldots$$

Take $\int_T^{2T} e^{-st} F(t)\, dt$

Let $t = u + T$ $\quad dt = du + 0$

when $t = T$,

$T = u + T$

$0 = u$

When $t = 2T$

$2T = u + T$

$T = u.$

$$\int_T^{2T} e^{-st} F(t)\, dt = \int_0^{T} e^{-s(u+T)} . F(u+T)\, du$$

$$= \int_0^{T} e^{-su} . e^{-sT} F(u) . du$$

$$= e^{-sT} \int_0^{T} e^{-su} . F(u)\, du$$

$$= e^{-sT} \int_0^{T} . e^{-st} . F(t) . dt$$

$$\int_{2T}^{3T} e^{-st} F(t)\, dt = e^{-2sT} \int_0^{T} . e^{-st} F(t)\, dt$$

$$\int_{3T}^{4T} e^{-st} F(t)\, dt = e^{-3sT} \int_0^{T} e^{-st} F(t)\, dt$$

$$\mathcal{L}\{F(t)\} = \int_0^{T} e^{-st} . F(t)\, dt + e^{-sT} \int_0^{T} e^{-st} F(t)\, dt + e^{-2sT} \int_0^{T} e^{-st} F(t) . dt \ldots$$

$$= \int_0^{T} e^{-st} F(t) . dt \left[ 1 + e^{-st} + e^{-2st} \ldots \right]$$

$$= \int_0^{T} e^{-st} F(t)\, dt \times \frac{1}{1 - e^{-st}}$$

$$= \frac{1}{1-e^{-sT}} \int_0^T e^{-st} F(t)\,dt$$

- If $F(t) = t^2 \quad 0<t<2$ and
  $F(t+2) = F(t) \quad t>2$. find $L(f(t))$

$$L[F(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} F(t)\,dt$$

$$= \frac{1}{1-e^{-s2}} \int_0^2 e^{-st} F(t)\,dt$$

$$= \frac{1}{1-e^{-2s}} \left( \left[ t^2 \cdot \frac{1}{-s} e^{-st} \right]_0^2 - \int_0^2 \frac{e^{-st}}{-s} \cdot 2t\,dt \right)$$

$$= \frac{1}{1-e^{-2s}} \left[ \frac{-4}{s} e^{-2s} - \frac{2}{-s} \left[ t \cdot \frac{e^{-st}}{-s} - \int_0^2 \frac{e^{-st}}{-s} \right]_0^2 \right]$$

$$= \frac{1}{1-e^{-2s}} \left[ \frac{-4}{s} e^{-2s} - \frac{4e^{-2s}}{s^2} + \frac{2}{s^3} e^{-3s} + \frac{2}{s^3} \right]$$

- For the periodic function $F(t)$ of period 4 defined by $F(t) = 3t \quad 0<t<2$
  $6 \quad 2<t<4$

$$L\{F(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} \cdot F(t)\,dt$$

$$= \frac{1}{1-e^{-s4}} \int_0^4 e^{-st} \cdot F(t)\,dt$$

$$= \frac{1}{1-e^{-4s}} \left[ \int_0^2 e^{-st} F(t)\,dt + \int_2^4 e^{-st} F(t)\,dt \right]$$

$$= \frac{1}{1-e^{-4s}} \left[ \int_0^2 e^{-st} 3t\,dt + \int_2^4 6e^{-st}\,dt \right]$$

$$= \frac{1}{1-e^{-4s}} \left[ 3\left[ t\frac{e^{-st}}{-s} - \int_0^2 \frac{e^{-st}}{-s}\,dt \right]_0^2 + 6\int_2^4 e^{-st}\,dt \right]$$

$$= \frac{1}{1-e^{-4s}} \left[ \frac{\cancel{2} e^{-2s}}{-s} - 3\left[\frac{e^{-2s}}{s^2}\right] - \frac{3}{s^2} + \frac{6}{-s}\left[e^{-4s} - e^{-2s}\right] \right]$$

$$= \frac{1}{1-e^{-4s}} \left[ -\frac{3}{s^2} e^{-2s} - \frac{3}{s^2} + \frac{6}{s} e^{-4s} \right]$$

$$= \frac{1}{1-e^{-4s}} \left[ -\frac{3}{s^2}\left[e^{-2s} - 1\right] - \frac{6}{s} e^{-4s} \right]$$

→ A periodic function of period $\frac{2\pi}{\omega}$ is defined by $f(t) = E\sin\omega t \quad 0 < t < \pi/\omega$

$= 0 \quad \pi/\omega < t < \frac{2\pi}{\omega}$

where E and ω are positive constants

$T = \frac{2\pi}{\omega}$

$$\mathcal{L}(F(t)) = \frac{1}{1-e^{-st}} \int_0^T e^{-st} F(t)\, dt$$

$$= \frac{1}{1-e^{-st}} \int_0^{2\pi/\omega} e^{-st} F(t)\, dt$$

$$= \frac{1}{1-e^{-st}} \left[ \int_0^{\pi/\omega} e^{-st} E\sin\omega t\, dt + \int_{\pi/\omega}^{2\pi/\omega} e^{-st} \cdot 0\, dt \right]$$

$$= \frac{E}{1-e^{-s\frac{2\pi}{\omega}}} \int_0^{\pi/\omega} e^{-st} \sin\omega t\, dt.$$

Let $I = \int_0^{\pi/\omega} e^{-st} \sin\omega t\, dt.$

$$= \left[ e^{-st} \frac{-\cos\omega t}{\omega} \right]_0^{\pi/\omega} - \int_0^{\pi/\omega} -\frac{1}{\omega} \cos\omega t \cdot e^{-st} \cdot -s\, dt$$

$$= \frac{-1}{\omega} \left( e^{-\pi/\omega s} ((-1)) - 1 \right) - \frac{s}{\omega} \int_0^{\pi/\omega} \cos\omega t \cdot e^{-st}\, dt$$

$$= \frac{1}{\omega} \left[ e^{-s\pi/\omega} + 1 \right] - \frac{s}{\omega} \left( \left[ e^{-st} \frac{1}{\omega} \sin\omega t \right]_0^{\pi/\omega} - \int_0^{\pi/\omega} e^{-st} \cdot -s\, \frac{1}{\omega} \sin\omega t \right)$$

$$= \left[ e^{-s\pi/\omega} + 1 \right] \frac{1}{\omega} - \frac{s}{\omega} \left[ 0 + \frac{s}{\omega} \int_0^{\pi/\omega} e^{-st} \sin\omega t \right]$$

$$= \frac{1}{\omega} \left( e^{-s\pi/\omega} + 1 \right] - \frac{s^2}{\omega^2} I$$

$$I \left[ 1 + \frac{s^2}{\omega^2} \right] = \frac{1}{\omega} \left[ e^{-s\pi/\omega} + 1 \right]$$

$$I = \frac{\omega}{\omega^2 + s^2} \left[ 1 + e^{-s\pi/\omega} \right]$$

$$\mathcal{L}[f(t)] = \frac{E}{1-e^{-st}} \left( \frac{\omega}{\omega^2+s^2} \cdot \left[ 1 + e^{-s\pi/\omega} \right] \right)$$

$$= \frac{E}{1-e^{-\frac{2s\pi}{\omega}}} \left[ \frac{\omega}{\omega^2+s^2} \left[ 1 + e^{-s\pi/\omega} \right] \right]$$

$$= \frac{E}{\left[ 1 - e^{-s\pi/\omega} \right] \cancel{\left[ 1 + e^{-s\pi/\omega} \right]}} \frac{\omega}{\omega^2+s^2} \cancel{1 + e^{-s\pi/\omega}}$$

$$= \frac{E\omega}{(1 - e^{-s\pi/\omega})(\omega^2+s^2)}$$

$\rightarrow F(t) = a$ for $0 \leq t < a$

$= -a$ for $a \leq t \leq 2a.$

$$L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2as}} \left[ \int_0^a e^{-st}\, a\, dt + \int_a^{2a} e^{-st}\, -a\, dt \right]$$

$$= \frac{a}{1-e^{-2as}} \left( \left[ \frac{e^{-st}}{-s} \right]_0^a + \left[ \frac{e^{-st}}{s} \right]_a^{2a} \right)$$

$$= \frac{a}{1-e^{-2as}} \left[ \frac{e^{-as}}{-s} + \frac{1}{s} + \frac{e^{-2as}}{s} - \frac{e^{-as}}{s} \right]$$

$$= \frac{a}{[1-e^{-2as}]s} \left[ 1 - e^{-as} + e^{-2as} - e^{-as} \right]$$

$$= \frac{a}{s[1-e^{-2as}]} \left[ 1 - 2e^{-2as} + e^{-2as} \right]$$

$$= \frac{a}{s[1-e^{-2as}]} \left[ 1 - e^{-as} \right]^2$$

$$= \frac{a\,[1-e^{-as}]^2}{s[1+e^{-as}]\,[1-e^{-as}]}$$

$$= \frac{a}{s} \, \frac{1-e^{-as}}{1+e^{-as}}$$

Q. $F(t) = e^t \quad 0 < t < 1$

$LF(t)$

NOTE: $$\frac{1-e^{-as}}{1+e^{-as}} = \frac{e^{-as/2}\left[e^{as/2} - e^{-as/2}\right]}{e^{-as/2}\left[e^{as/2} + e^{-as/2}\right]}$$

$$= \frac{\sinh \frac{as}{2}}{\cosh \frac{as}{2}} = \tanh \frac{as}{2}$$

Q. $F(t) = e^t \quad 0 < t < 1$

$$L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} F(t)\, dt$$

$$= \frac{1}{1-e^{-s}} \int_0^1 e^{-(s+1)t}\, dt$$

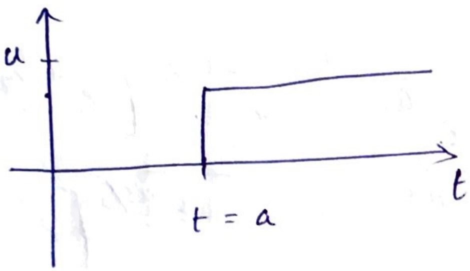
$$= \frac{1}{1-e^{-s}} \cdot \frac{1}{s+1} \left[ 1 - e^{-s+1} \right]$$

UNIT STEP FUNCTION [HEAVISIDE FUNCTION]

A unit step function is defined by $u[t-a]=0$ or $t \leq a$

$=1$ for $t>1$

Graph of figure is as follows.




It is also denoted by $H[t-a]$, therefore we get

$H(t-a)=0$ for $t \leq a$

$=1$ for $t>a$

→ Find laplace transform of unit step func.

$$L[u(t-a)] = \int_0^\infty e^{-st} u(t-a) dt$$

$$= \int_0^a e^{-st} 0 . dt + \int_a^\infty e^{-st} . dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_a^\infty$$

$$= \frac{1}{-s}\left[0 - e^{-as}\right] = \frac{1}{s} e^{-as}$$

SECOND SHIFTING PROPERTY [HEAVISIDE SHIFTING PROPERTY]

If $f(t)$ is any function of $t$ then,

$L[f(t-a).H(t-a)] = e^{-as}.L[f(t)]$

$H[t-a]=0$ for $t \leq a$

$=1$ for $t>a$

$$L[F(ta)H(t-a)] = \int_0^\infty e^{-st}.F(t-a)H(t-a).dt.$$

$$= \int_0^a e^{-st} F(t-a)(0)dt + \int_a^\infty e^{-st}.F(t-a).dt)$$

Put $t-a=u$

$dt = du$

when $t=a$

$u=0$

when $t=\infty$ $u=\infty$

$$L[F(t-a)H(t-a)] = \int_0^\infty e^{-st} . F(u) . du$$

$$= \int_0^\infty e^{-s(u+a)} . F(u) . du$$

$$= \int_0^\infty e^{-su} . e^{-as} . F(u)\, du$$

$$= -e^{-as} \int_0^\infty e^{-su} . F(u)\, du.$$

$$= e^{-as} \int_0^\infty e^{-st} f(t)\, dt$$

$$= e^{-as} L[F(t)]$$

• Find laplace transform of the following functions $[[e^{(t-1)} H(t-a)]$

$$L[e^{(t-1)} . H(t-1)] = e^{-s} L[e^{(t)}]$$

$$= e^{-s} \int_0^\infty e^{t}\, dt$$

$$= e^{-s} . \frac{1}{s-1}$$

$$\Rightarrow L[t^2 . H(t-3)]$$

$$= L[(t-3+3)^2 . H(t-3)]$$

$$= L[((t-3)^2 + 6(t-3) + 9] . H(t-3)]$$

$$= L[(t-3)^2 . H(t-3)] + 6L[(t-3)H(t-3)] + 9L(H(t-3))]$$

$$= e^{-3s} L(t^2) + 6 e^{-3s} L(t) + 9 \frac{e^{-3s}}{s}$$

$$= \frac{e^{-3s} . 2!}{s^{2+1}} + \frac{6e^{-3s}}{s^2} + \frac{9e^{-3s}}{s}$$

$$= e^{-3s}\left[\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right]$$

• $L[e^{-t} H(t-2)]$

$$= L[e^{-(t+2-2)} H(t-2)]$$

$$= L[e^{-(t-2)-2} . H(t-2)]$$

$= L\left[e^{-(t-2)} \cdot e^{-2} \cdot H(t-2)\right]$

$= e^{-2} \cdot L\left[e^{-(t-2)} \cdot H(t-2)\right]$

$= \frac{1}{e^2} e^{-2s} L(e^{-t})$

$= \frac{e^{-2s}}{e^2} \cdot \frac{1}{s+1}$

Q. $L\left[(1-e^{2t}) H(t-2)\right]$

$= L[H(t-2)] - L\left[e^{2t} \cdot H(t-2)\right]$

$= \frac{e^{-2s}}{s} - L\left[e^{2(t-2+2)} H(t-2)\right]$

$= \frac{e^{-2s}}{s} - L\left[e^{2(t-2)} \cdot e^{4} \cdot H(t-2)\right]$

$= \frac{e^{-2s}}{s} - e^4 L\left[e^{2(t-2)} \cdot H(t-2)\right]$

$= \frac{e^{-2s}}{s} - e^4 \cdot e^{-2s} \cdot \frac{1}{s-2}$

$= e^{-2s}\left[\frac{1}{s} - \frac{e^4}{s-2}\right]$

Q. $F(t) = F_1(t) \quad t \leq a$

$= F_2(t) \quad t > a$

$F(t) = F_1(t) + \left[F_2(t) - F_1(t)\right] \cdot H(t-a)$

Proof:

$H(t-a) = 0$ for $t \leq a$

$= 1$ for $t > a$

$(F_2(t) - F_1(t))\, H(t-a) = 0$ if $t \leq a$

$= F_2(t) - F_1(t) \quad t > a$

Adding $F_1(t)$ on both sides

$F_1(t) + \left[F_2(t) - F_1(t)\right] H(t-a)$

$= 0 + F_1(t)$ for $t \leq a$

$= F_2(t) - F_1(t) + F_1(t)$ if $t > a$

$= F_1(t)$ if $t \leq a$

$= F_2(t)$ if $t > a$

$F_1(t) + \left[F_2(t) - F_1(t)\right] H(t-a) = F(t)$

Express the following functions in terms of unit step functions and hence find the Laplace transform.

$F(t) = 2t \quad 0 < t < \pi$

$= 1 \quad \text{for } t > \pi$

$F(t) = 2t + (1-2t)\, H(t-\pi)$

$= 2t + H(t-\pi) - 2t\,(H(t-\pi)$

$= 2t + H(t-\pi) - 2\cancel{(t-\pi)H(t-\pi)} - \cancel{2\pi H(t-\pi)}$

$2((t-\pi+\pi)\,H(t-\pi))$

$= 2t + H(t-\pi) - 2(t-\pi)H(t-\pi) - 2\pi\, H(t-\pi)$

$L[F(t)] = L[2t] + L\,H(t-\pi) - 2\,L[(t-\pi)H(t-\pi)] - 2\pi\, L[H(t-\pi)]$

$= 2\cdot\frac{1}{s^2} + \frac{e^{-\pi s}}{s} - 2e^{-\pi s}L(t) - \frac{2\pi e^{-\pi s}}{s}$

$= \frac{2}{s^2} + \frac{e^{-\pi s}}{s} - 2e^{-\pi s}\cdot\frac{1}{s^2} - \frac{2\pi e^{-\pi s}}{s}$

$\Rightarrow F(t) = 3t \quad \text{for } 0 < t < 2$

$= 6 \quad \text{for } 2 < t < 4$

$T = 4$

$F(t) = 3t + (6-3t)H(t-2)$

$= 3t + 6H\cdot(t-2) - 3t\,H(t-2)$

$= 3t + 6H(t-2) - 3(t-2+2)H(t-2)$

$= 3t + 6H(t-2) - 3(t-2)H(t-2) - 6H(t-2)$

$= 3t + 6H(t-2) - 3(t-2)H(t-2) - 6H(t-2)$

$L(F(t)) = 3L(t) - 3L((t-2)H(t-2))$

$= 3\frac{1}{s^2} - 3e^{2s}L(t)$

$= \frac{3}{s^2} - 3e^{2s}\frac{1}{s^2}$

$= \frac{3}{s^2}\left[1 - e^{2s}\right]$

Q. $F(t) = t^2 \quad 1 < t < 2$

$= 4t \quad \text{for } t \geq 2 \qquad T =$

$F(t) = t^2 + (4t - t^2)\,H\cdot(t-2)$

$= t^2 + 4t\,H(t-2) - t^2 H(t-2)$

$$= t^2 + 4\left[(t-2+2)H(t-2) - (t-2+2)^2 H(t-2)\right.$$

$$= t^2 + 4[t-2]H[t-2] + 8H(t-2) -$$

$$[(t-2)^2 + 4(t-2) + 4]H(t-2)$$

$$= t^2 + 4[t-2]H(t-2] + 8H(t-2) - (t-2)H(t-2)$$

$$-4(t-2)H(t-2) - 4H(t-2)$$

$$= t^2 - (t-2)^2 H(t-2) + 4H(t-2)$$

$$L[F(t)] = L(t)^2 - L(t-2)^2 H(t-2) + 4LH(t-2)$$

$$= \frac{2!}{s^{2+1}} - e^{2t} L(t)^2 + 4e^{2t} 1/s$$

$$= \frac{2}{s^3} - e^{2t}\frac{2}{s^3} + 4\frac{e^{2t}}{s}$$

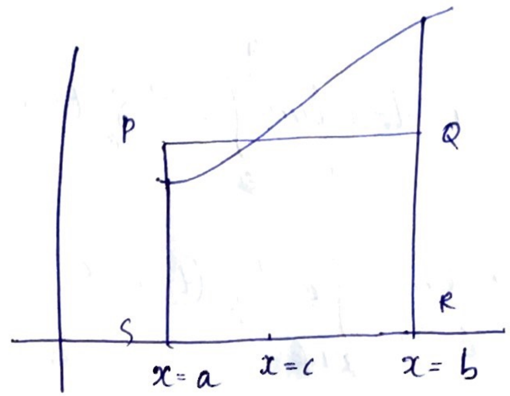
$$L([1-e^{2t}]H[t-2])$$

1. The mean value theorem on integration

The area under the curve $y = f(x)$ and $x$ axis from $x = a$ to $x = b$

$$\int_a^b f(x)\,dx = (b-a)f(c)$$

where $a < c < b$




Area of PQRS $= (b-a)f(c)$ $\quad a < c < b$

## IMPULSE FUNCTION.

An impulse fn is defined as

$$\delta[t-a] = \lim_{\epsilon \to 0} \delta_\epsilon(t-a]$$

where $\delta_\epsilon = \frac{1}{\epsilon}$ for $a < t < a+\epsilon$

$= 0$ otherwise

Find laplace transform of the impulse function

Proof:

$$L(\delta(t-a)) = \lim_{\epsilon \to 0} L(\epsilon(t-a)$$

$$= \lim_{\epsilon \to 0} \int_0^\infty e^{-st} \cdot \delta(t-a) \cdot dt$$

$$= \lim_{\epsilon \to 0} \int_0^a e^{-st} \delta_\epsilon (t-a)\, dt + \lim_{\epsilon \to 0} \int_a^{a+\epsilon} e^{-st} \cdot \delta_\epsilon (t-a)\, dt$$

$$+ \lim_{\epsilon \to 0} \int_{a+\epsilon}^\infty e^{-st} \delta_\epsilon (t-a)\, dt$$

$$= 0 + \lim_{\epsilon \to 0} \int_a^{a+\epsilon} e^{-st} \cdot \frac{1}{\epsilon}\, dt + 0$$

when $F(t) = e^{-st}$

$$F(\eta) = e^{-s\eta}$$

$$= \lim_{\epsilon \to 0} \frac{1}{\epsilon} (a + \epsilon - a) f(\eta)$$

when $a < \eta < a + \epsilon$

$\lim_{\epsilon \to 0} a < \eta < a + \epsilon$

$$= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \times \epsilon \cdot e^{-s\eta}$$

$\Rightarrow a < \eta < a + \epsilon$

$$= e^{-sa}$$

## CONVOLUTION :

When $t > 0$ and $f(t)$ and $g(t)$ are any 2 functions of $t$, then the convolution between two functions is defined as

$$f(t) * g(t) = \int_0^t f(u)\, g(t-u)\, du$$

$$= \int_0^t g(u)\, f(t-u) \cdot du$$

Convolution theorem in Laplace transform

If $f(t)$ and $g(t)$ are any 2 functions of $(t)$, then $\mathcal{L}[F(t)]\mathcal{L}(G(t)) = \mathcal{L}(F(t) * G(t))$

* Verify the lap convolution theorem of the function $f(t) = t$ & $G(t) = e^t$

$\mathcal{L}(F(t)\, \mathcal{L}(G(t)) = \mathcal{L}(F(t) * G(t))$

$\mathcal{L}(F(t)) = L(t) = \frac{1}{s^2}$

$\mathcal{L}(G(t)) = L(e^t) = \frac{1}{s-1}$

$\mathcal{L}(F(t))\, L(G(t)) = \frac{1}{s^2(s-1)}$

$$F(t) * G(t) = \int_0^t F(u)\, G(t-u)\, du$$

$$= \int_0^t u \cdot e^{(t-u)} \cdot du$$

$$= \int_0^t u \cdot e^t \cdot e^{-u} \cdot du$$

$$= e^t \int_0^t u\, e^{-u} \cdot du$$

$$= e^t \left[ \left( u \cdot \frac{1}{-1} e^{-u} \right)_0^t - \int_0^t \frac{1}{-1} e^{-u} \cdot 1 \cdot du \right]$$

$$= e^t \left[ -(t \cdot e^{-t} - 0) + \frac{1}{-1} \left. e^{-u} \right]_0^t \right]$$

$$= e^t \left[ -t e^{-t} - (e^{-t} - e^0) \right]$$

$$= e^t \left[ -t e^{-t} - e^{-t} + 1 \right]$$

$$= -t - 1 + e^t$$

$$L[F(t) * G(t)] = -L(t) - L(1) + L(e^t)$$

$$= -\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s-1}$$

$$= \frac{(s-1) - s(s-1) + s^2}{s^2(s-1)}$$

$$= \frac{s - 1 - s^2 + s + s^2}{s^2(s-1)}$$

$$= \frac{1}{s^2(s-1)}$$

$$L(F(t)) L[G(t)] = L[F(t) * G(t)]$$

* Verify Convolution theorem of the function $F(t) = 1$ and $G(t) = \sin t$

$$L(F(t)) = L(1) = \frac{1}{s}$$

$$L(G(t)) = L(\sin t) = \frac{1}{s^2 + 1^2}$$

$$L(F(t)) L(G(t)) = \frac{1}{s} \cdot \frac{1}{s^2 + 1^2}$$

$$= \frac{1}{s(s^2+1)}$$

$$F(t) * G(t) = \int_0^t F(u) \cdot G(t-u) \cdot du$$

$$= \int_0^t 1 \cdot \sin(t-u)\, du$$

$$= \int_0^t \left[ \frac{-\cos(t-u)}{-u} \right]$$

$= \cos(t-t) - \cos(t-0)$

$= \cos 0 - \cos t$

$= 1 - \cos t$

$\mathcal{L}(F(t) * G(t)) = L(1-\cos t)$

$= L(1) - \mathcal{L}(\cos t)$

$= \frac{1}{s} - \frac{s}{s^2+1}$

$= \frac{s^2+1-s^2}{(s^2+1)s}$

$= \frac{1}{s(s^2+1)}$

$\therefore L[(F(t) * G(t)] = L(f(t)) \, L(G(t))$

EVALUATION OF THE DEFINITE INTEGRAL USING LAPLACE TRANSFORM

Evaluate $\int_0^\infty e^{-st} \cdot t^3 \cdot \cos t \, dt$

$= \mathcal{L}(t^3 \cos t)$

$= (-1)^3 \frac{d^3}{ds^3} \mathcal{L}(\cos t)$

$= -\frac{d^3}{ds^3} \frac{s}{s^2+1^2}$

$= -\frac{d^2}{ds^2} \cdot \frac{d}{ds} \frac{s}{s^2+1}$

$= -\frac{d^2}{ds^2} \frac{(s^2+1) - s(2s)}{(s^2+1)^2}$

$= \frac{-d^2}{ds^2} \frac{(1-s^2)}{(s^2+1)^2}$

$= \frac{-d}{ds} \cdot \frac{d}{ds} \frac{(1-s^2)}{(s^2+1)^2}$

$= \frac{-d}{ds} \frac{(s^2+1)^2(-2s) - (1-s^2)2(s^2+1)2s}{(s^2+1)^4}$

$= \frac{-d}{ds} \left[ \frac{2s[s^2+1][-(s^2+1) - 2(1-s^2)]}{(s^2+1)^4} \right]$

$= -2 \frac{d}{ds} \frac{s[-s^2-1-2+2s^2]}{(s^2+1)^3}$

$$= -2\frac{d}{ds}\,\frac{s(s^2-3)}{(s^2+1)^3}$$

$$= -2\frac{d}{ds}\,\frac{s^3-3s}{[s^2+1]^3}$$

$$= -2\frac{\left[(s^2+1)^3(3s^2-3)-(s^3-3s)\,3(s^2+1)^2\,2s\right]}{(s^2+1)^6}$$

$$= -2\frac{(s^2+1)^2\left[(s^2+1)(3s^2-3)-6s(s^3-3s)\right]}{(s^2+1)^6}$$

$$= -2\frac{\left[3s^4-3s^2+3s^2-3-6s^4+18s^2\right]}{(s^2+1)^6}$$

$$= -2\frac{\left[18s^2-3s^4-3\right]}{(s^2+1)^4}$$

$$= \frac{6\left[s^4+1-6s^2\right]}{(s^2+1)^4}$$

$$\int_0^\infty e^{-3t}\cdot t\cdot \sin t\cdot dt$$

Put $s=3$

$$I = \int_0^\infty e^{-st}\, t\sin t\, dt$$

$$= L(t\sin t)$$

$$= -1\frac{d}{ds}L(\sin t)$$

$$= -1\frac{d}{ds}\frac{1}{s^2+1^2}$$

$$= -\times -\frac{1}{(s^2+1)^2}\cdot 2s$$

$$= \frac{2s}{(s^2+1)^2} = \frac{2(3)}{(3^2+1^2)^2} = \frac{6}{100} = \frac{3}{50}$$

$$\int_0^\infty t^3 e^{-t}\sin t\, dt$$

Put $s=1$

$$I = \int_0^\infty e^{-st} t^3 \sin t\, dt$$

$$= L(t^3 \sin t)$$

$$= (-1)^3\frac{d^3}{ds^3}L(\sin t) = -\frac{d^3}{ds^3}\frac{1}{s^2+1}$$

$$= -\frac{d^2}{ds^2}\left[\frac{-1(2s)}{(s^2+1)^2}\right]$$

$$= 2\frac{d^2}{ds^2}\left[\frac{s}{(s^2+1)^2}\right]$$

$$= 2\frac{d}{ds}\frac{(s^2+1)^2 \cdot 1 - s\, 2(s^2+1)(2s)}{(s^2+1)^4}$$

$$= 2\frac{d}{ds}\frac{(s^2+1)\left[(s^2+1) - 4s^2\right]}{(s^2+1)^4}$$

$$= 2\frac{d}{ds}\frac{(1-3s^2)}{(s^2+1)^3}$$

$$= 2\frac{\left[(s^2+1)^3 \cdot -6s - (1-3s^2)\, 3(s^2+1)^2(2s)\right]}{(s^2+1)^6}$$

$$= \frac{2(s^2+1)^2\left[-6s\cdot(s^2+1) - 6s(1-3s^2)\right]}{(s^2+1)^6}$$

$$= 2\frac{\left[-6s(s^2+1) - 6s(1-3s^2)\right]}{(s^2+1)^4}$$

$$= -12s\left[\frac{s^2+1+1-3s^2}{(s^2+1)^4}\right]$$

$$= 12\left[\frac{1+1+1-3}{(1+1)^4}\right]$$

$$= \underline{\underline{0}}$$

$$\Rightarrow \int_0^\infty t \cdot e^{-2t} \sin^2 t \, dt$$

Put $s=2$

$$I = \int_0^\infty \cdot e^{-st}\, t \sin^2 t \, dt$$

$$= L(t \sin^2 t)$$

$$= (-1)\frac{d}{ds} L(\sin^2 t)$$

$$= -\frac{d}{ds}\left(\tfrac{1}{2} - \tfrac{1}{2}\cos 2t\right)$$

$$= -\frac{d}{ds}\left(\tfrac{1}{2} L(1) - \tfrac{1}{2} L(\cos 2t)\right)$$

$$= -\tfrac{1}{2}\frac{d}{ds}\left(\frac{1}{s} - \frac{s}{s^2+2^2}\right)$$

$$= -\tfrac{1}{2}\left[-\tfrac{1}{s^2} - \left(\frac{(s^2+4) - 2(2s)}{(s^2+4)^2}\right)\right]$$

$$= \tfrac{1}{2}\left[\tfrac{1}{s^2} + \frac{4-s^2}{[s^2+4]^2}\right]$$

$$= \tfrac{1}{2}\left[\tfrac{1}{4} + \frac{4-4}{(4+4)^2}\right]$$

$$= \tfrac{1}{8}$$

$\Rightarrow \int_0^\infty \frac{e^{-t}\sin t}{t}\,dt \qquad s=1$

$$I = \int_0^\infty e^{-st}\frac{\sin t}{t}\,dt$$

$$= L\left(\frac{\sin t}{t}\right)$$

$$= \int_s^\infty L(\sin t)\,ds$$

$$= \int_s^\infty \frac{1}{s^2+1}\,ds$$

$$= \tan^{-1}s\Big]_s^\infty$$

$$= \tan^{-1}\infty - \tan^{-1}s$$

$$= \pi/2 - \tan^{-1}s$$

$$= \cot^{-1}s = \cot^{-1}1 = \pi/4$$

Q. $\int_0^\infty \frac{\sin mt}{t}\,dt$

$$= \int_0^\infty e^{0t}\,\frac{\sin mt}{t}\,dt \qquad s=0$$

$$I = \int_0^\infty e^{-st}\,\frac{\sin mt}{t}\,dt$$

$$= L\left(\frac{\sin mt}{t}\right)$$

$$= \int_s^\infty L(\sin mt)\,ds$$

$$= \int_s^\infty \frac{m}{s^2+m^2}\,ds$$

$$= m\int_s^\infty \frac{1}{s^2+m^2}\,ds$$

$$= m\,\tfrac{1}{m}\tan^{-1}\tfrac{s}{m}\Big]_s^\infty$$

$= \tan^{-1}\infty - \tan^{-1} s/m$

$= \pi/2 - \tan^{-1} s/m = \cot^{-1} s/m = \cot^{-1} 0$

$= \pi/2$

$\rightarrow I = \int_0^{\infty} \frac{e^{-t} - e^{-3t}}{t} \cdot dt$

$= \int_0^{\infty} e^{-0t} \cdot \frac{(e^{-t} - e^{-3t})}{t} \cdot dt$

Put $s = 0$

$I = \int_0^{\infty} e^{-st} \left(\frac{e^{-t} - e^{-3t}}{t}\right) dt$

$= L\left(\frac{e^{-t} - e^{-3t}}{t}\right)$

$= \int_s^{\infty} L(e^{-t} - e^{-3t}) ds$

$= \int_s^{\infty} \left(\frac{1}{s+1} - \frac{1}{s+3}\right) ds$

$= \Big|\log(s+1) - \log(s+3)\Big|_s^{\infty}$

$= \log \frac{s+1}{s+3}\Big|_s^{\infty}$

$= \log \left.\frac{s(1+1/s)}{s(1+3/s)}\right]_s^{\infty}$

$= \log \frac{1+1/\infty}{1+3/\infty} - \log \frac{s+1}{s+3}$

$= \log 1 - \log \frac{s+1}{s+3}$

$= \log \frac{s+3}{s+1}$

$= \log \frac{0+3}{0+1} = \log 3$

INVERSE TRANSFORM

$L(1) = \frac{1}{s}$

1. $L^{-1} \frac{1}{s} = 1$

$L(t) = \frac{1}{s^2}$

2. $L^{-1} \frac{1}{s^2} = t$

$$\mathcal{L}(t^n) = \frac{\lfloor n}{s^{n+1}}$$

$$t^n = L^{-1}\frac{\lfloor n}{s^{n+1}}$$

$$L^{-1}\frac{1}{s^{n+1}} = \frac{t^n}{\lfloor n}$$

3. $$L^{-1}\frac{1}{s^n} = \frac{t^{n-1}}{\lfloor n-1}$$

$$= \frac{t^{n-1}}{\overline{\lfloor n}}$$

$$L(e^{at}) = 1/s-a$$

4. $L^{-1}\, 1/s-a = e^{at}$

5. $L^{-1}\, 1/s+a = e^{-at}$

$$L(\sin at) = \frac{a}{s^2+a^2}$$

$$\sin at = L^{-1}\frac{a}{s^2+a^2}$$

5 $$\mathcal{L}^{-1}\frac{1}{s^2+a^2} = \frac{1}{a}\sin at$$

$$\mathcal{L}\cos at = s/s^2+a^2$$

6. $$\mathcal{L}^{-1}\frac{s}{s^2+a^2} = \cos at$$

$$L[\sinh at] = \frac{a}{s^2-a^2}$$

$$\sinh at = \mathcal{L}^{-1}\frac{a}{s^2-a^2}$$

$$\frac{1}{a}\sinh at = \mathcal{L}^{-1}\frac{1}{s^2-a^2}$$

7. $$\mathcal{L}^{-1}\frac{1}{s^2-a^2} = \frac{1}{a}\sinh at$$

$$\mathcal{L}\cosh at = \frac{s}{s^2-a^2}$$

8. $$\mathcal{L}^{-1}\frac{s}{s^2-a^2} = \cosh at$$

9. $$\mathcal{L}^{-1}\frac{s}{(s^2+a^2)^2} = \frac{1}{2a}t\sin at$$

10. $$\mathcal{L}^{-1}\frac{1}{(s^2+a^2)^2} = \frac{1}{2a^3}[\sin at - at\cos at]$$

# DIFFERENT METHODS OF FINDING INVERSE ~~LAPLACES~~ TRANSFORM

## I. By Direct Method

1. Find inverse transform of following

a) $$L^{-1}\frac{2}{s} = 2L^{-1}\frac{1}{s}$$
$$= 2\times 1 = \underline{2}$$

b) $$\mathcal{L}^{-1}\frac{3}{s+3} = 3\mathcal{L}^{-1}\frac{1}{s+3}$$
$$= \underline{3e^{-3t}}$$

c) $$\mathcal{L}^{-1}\frac{2s}{s^2+4} = 2\mathcal{L}^{-1}\frac{s}{s^2+2^2} = \underline{2\cos 2t}$$

d) $$\mathcal{L}^{-1}\frac{1}{2s^2+9} = \frac{1}{2}\mathcal{L}^{-1}\frac{1}{s^2+\frac{9}{2}} = \frac{1}{2}\mathcal{L}^{-1}\frac{1}{s^2+\left(\frac{3}{\sqrt{2}}\right)^2}$$
$$= \frac{1}{2}\times\frac{1}{3/\sqrt{2}}\sin\frac{3}{\sqrt{2}}t$$
$$= \underline{\frac{1}{3\sqrt{2}}\sin 3/\sqrt{2}\, t}$$

e) $$\mathcal{L}^{-1}\frac{2s-1}{s^2+8} = \mathcal{L}^{-1}\frac{2s}{s^2+(\sqrt{8})^2} - \mathcal{L}^{-1}\frac{1}{s^2+(\sqrt{8})^2}$$
$$= 2\mathcal{L}^{-1}\frac{s}{s^2+(\sqrt{8})^2} - \mathcal{L}^{-1}\frac{1}{s^2+(\sqrt{8})^2}$$

$= 2\cos^{-1}\sqrt{8}t - \frac{1}{\sqrt{8}}\sin\sqrt{8}t$

f) $L^{-1}\frac{3s-4}{16-s^2}$

$= 3L^{-1}\frac{s}{16-s^2} - 4L^{-1}\frac{1}{16-s^2}$

$= -3L^{-1}\frac{s}{s^2-4^2} + 4L^{-1}\frac{1}{s^2-4^2}$

$= -3\cosh 4t + 4\times\frac{1}{4}\sinh 4t$

$= -3\cosh 4t + \sinh 4t$

g) $L^{-1}\frac{3s^2+4}{s^5}$

$= 3L^{-1}\frac{s^2}{s^5} + 4L^{-1}\frac{1}{s^5}$

$= 3L^{-1}\frac{1}{s^3} + 4L^{-1}\frac{1}{s^5}$

$= 3\times\frac{1}{\lfloor 3-1}.t^{3-1} + 4\frac{1}{\lfloor 5-1}.t^{5-1}$

$= \frac{3}{2}t^2 + \frac{1}{6}t^4$

h) $L^{-1}\frac{1}{s^{5/2}}$

$= \frac{1}{\Gamma 5/2}t^{5/2-1}$

$= \frac{1}{3/2\,\Gamma 3/2}.t^{3/2}$

$= \frac{1}{3/2\times\frac{1}{2}\Gamma\frac{1}{2}}.t^{3/2}$

$= \frac{1}{3/4\sqrt{\pi}}\times 3/2$

$= \frac{4}{3\sqrt{\pi}}$

i) $L^{-1}\frac{3(s^2-2)^2}{2s^5}$

$= \frac{3}{2}L^{-1}\left(\frac{s^4-4s^2+4}{s^5}\right)$

$= \frac{3}{2}L^{-1}\left(\frac{s^4}{s^5} - \frac{4s^2}{s^5} + \frac{4}{s^5}\right)$

$= \frac{3}{2}L^{-1}\left[\frac{1}{s} - \frac{4}{s^3} + \frac{4}{s^5}\right]$

$= \frac{3}{2}\left(1 - 4\times\frac{1}{\lfloor 3-1}t^{3-1} + 4\frac{1}{\lfloor 5-1}t^{5-1}\right)$

$= 3/2\left[1 - 2t^2 + 1/6\,t^4\right]$

j] $L\left[\frac{s^2+as+b}{s^3}\right]$

$L^{-1}\left[\frac{s^2}{s^3}+\frac{as}{s^3}+\frac{b}{s^3}\right]$

$= L^{-1}\frac{1}{s} + aL^{-1}\frac{1}{s^2} + bL^{-1}\frac{1}{s^3}$

$= 1 + at + b\frac{1}{\lfloor 3-1}t^2$

$= 1+at+\frac{bt^2}{2}$

k) $L^{-1}\frac{s}{2s^2-8}$

$= L^{-1}\frac{s}{2(s^2-4)}$

$= \frac{1}{2}L^{-1}\frac{s}{s^2-4}$

$= \frac{1}{2}\cosh 2t$

l) $L^{-1}\frac{1}{9s^2+1}$

$= \frac{1}{9}L^{-1}\frac{1}{s^2+1/9}$

$\frac{1}{9}$ $\frac{1}{s^2+(1/3)^2}$

$= \frac{1}{3}\sin \frac{1}{3}t$

m] $L^{-1}\frac{s+b}{s^2+a^2}$

$= L^{-1}\frac{s}{s^2+a^2} + bL^{-1}\frac{1}{s^2+a^2}$

$= \cos at + \frac{b}{a}\sin at$

n] $L^{-1}\left[\frac{2s-5}{4s^2+25}+\frac{4s}{9-s^2}\right.$

$= L^{-1}\frac{2s}{4s^2+25} - \frac{5}{4s^2+25} + \frac{4s}{9-s^2}$
$\left.+\frac{18}{9-s^2}\right]$

$= \frac{1}{2}L^{-1}\frac{s}{s^2+\frac{25}{4}} - \frac{5}{4}L^{-1}\frac{1}{s^2+\frac{25}{4}}$

$- 4L^{-1}\frac{s}{s^2-3^2} - \frac{18}{s^2-9}$

$= \frac{1}{2}L^{-1}\frac{s}{s^2+\left(\frac{5}{2}\right)^2} - \frac{5}{4}\times\frac{1}{5/2}\sin 5t - 4L^{-1}\frac{s}{s^2-3^2} + 18L^{-1}\frac{1}{s^2-3^2}$

$= \frac{1}{2}\cos\frac{5}{2}t - \frac{1}{2}\sin 5t - 4\cosh 3t + 6\sinh 3t$

2) USING THE SHIFTING PROPERTY.

$L[F(t)] = F(s)$

Then $L[e^{at}F(t)] = f(s-a)$

$e^{at}F(t) = L^{-1}[F(s-a)]$

$L^{-1}[f(s-a)] = e^{at}F(t)$

$= e^{at}L^{-1}F(s)$

Thus $L^{-1}f(s+a) = e^{-at}L^{-1}f(s)$

a] $L^{-1}\frac{1}{s-a} = e^{at}L^{-1}\left(\frac{1}{s}\right)$

$= e^{at}\cdot 1$

b) $L^{-1}\frac{1}{s+a} = e^{-at}L^{-1}1/s$

$= e^{-at}\cdot 1$

$= e^{-at}$

c] $L^{-1}\frac{1}{(s-a)^n} = e^{at}L^{-1}\frac{1}{s^n}$

$= e^{at}\frac{1}{\lfloor n-1}t^{n-1}$

d) $L^{-1}\frac{1}{(s+a)^n} = e^{-at} L^{-1}\frac{1}{s^n}$

$= e^{-at} \cdot \frac{1}{\lfloor n-1} \cdot t^{n-1}$

e) $L^{-1}\frac{1}{(s-a)^2+b^2} = e^{at} \cdot L^{-1}\frac{1}{s^2+b^2}$

$= e^{at} \frac{1 \sin bt}{b}$

f) $L^{-1}\frac{s-a}{(s-a)^2+b^2} = e^{at} L^{-1}\frac{s}{s^2+b^2}$

$= e^{at}\cos bt$

g) $L^{-1}\frac{1}{(s+a)^2-b^2} = e^{-at} L^{-1}\frac{1}{s^2-b^2}$

$= e^{-at} \cdot \frac{1}{b} \sinh bt$

h) $L^{-1}\frac{s+a}{(s+a)^2-b^2} = e^{-at}\cosh bt$

i) $L^{-1}\frac{1}{(s-a)^2-b^2} = e^{at} L^{-1}\frac{1}{s^2-b^2}$

$= e^{at} \frac{1}{b} \sin bt$

j) $L^{-1}\frac{s-a}{(s-a)^2-b^2} = e^{at} L^{-1}\frac{s}{s^2-b^2} = e^{at}\cosh bt$

k) $L^{-1}\frac{s^2}{(s-a)^3}$

$= L^{-1}\frac{(s-a+a)^2}{(s-a)^3}$

$= L^{-1}\frac{(s-a)^2+2(s-a)a+a^2}{(s-a)^3}$

$= L^{-1}\frac{(s-a)^2}{(s-a)^3} + 2aL^{-1}\frac{(s-a)}{(s-a)^{3}} + L^{-1}\frac{a^2}{(s-a)^3}$

$= L^{-1}\frac{1}{s-a} + 2a L^{-1}\frac{1}{(s-a)^2} + a^2 L^{-1}\frac{1}{(s-a)^3}$

$= e^{at} + 2ae^{at} L\frac{1}{s^2} + a^2 e^{at} L^{-1}\frac{1}{s^3}$

$= e^{at} + 2a e^{at} t + a^2 e^{at} \cdot \frac{1}{\lfloor 3-1} t^{3-1}$

$= e^{at} + 2at e^{at} + \frac{a^2 e^{at}}{2} t^2$

l) $L^{-1}\frac{s+2}{s^2-4s+13}$

$= L^{-1}\frac{s-2+4}{s^2-4s+4+9} = L^{-1}\frac{(s-2)+4}{(s-2)^2+3^2}$

$$= \mathcal{L}^{-1} \frac{s-2}{(s-2)^2+3^2} + 4\mathcal{L}^{-1} \frac{1}{(s-2)^2+3^2}$$

$$= e^{2t} \mathcal{L}^{-1} \frac{s}{s^2+3^2} + 4 e^{2t} \mathcal{L}^{-1} \frac{1}{s^2+3^2}$$

$$= e^{2t} \cos 3t + \frac{4e^{2t}}{3} \sin 3t$$

m) $\mathcal{L}^{-1} \frac{3s+7}{s^2-2s-3} = \mathcal{L}^{-1} \frac{3s+7}{s^2-2s+1-4}$

$$= \mathcal{L}^{-1} \frac{3s+7}{(s-1)^2-2^2}$$

$$= \mathcal{L}^{-1} \frac{3(s-1+1)+7}{(s-1)^2-2^2}$$

$$= 3\mathcal{L}^{-1} \frac{s-1}{(s-1)^2-2^2} + \cancel{3\mathcal{L}^{-1} \frac{1}{(s-1)^2-2^2}} + 10\mathcal{L}^{-1} \frac{1}{(s-1)^2-2^2}$$

$$= 3e^t \mathcal{L}^{-1} \frac{s}{s^2-2^2} + 10 e^t \mathcal{L}^{-1} \frac{1}{s^2-2^2}$$

$$= 3e^t \cos 2t + \frac{10}{2} e^t \sin 2t$$

$$= 3e^t \cos 2t + 5e^t \sin 2t$$

n) $\mathcal{L}^{-1} \frac{s}{s^4+s^2+1}$

$$s^4+s^2+1 = s^4+2s^2+1-s^2$$

$$= (s^2+1)^2 - s^2$$

$$= (s^2+1+s)(s^2+1-s)$$

$$\frac{s}{s^4+s^2+1} = \frac{s}{(s^2+s+1)[s^2-s+1]}$$

$$= \frac{1}{2}\left[\frac{1}{s^2-s+1} - \frac{1}{s^2+s+1}\right] \times \frac{1}{2}$$

$$= \frac{1}{s^2-s+1} - \frac{1}{s^2+s+1}$$

$$= \frac{s^2+s+1-s^2+s-1}{[s^2-s+1][s^2+s+1]}$$

$$= \frac{2s}{[s^2-s+1](s^2+s+1)}$$

$=$

$$= \frac{1}{2}\left[\frac{1}{s^2-s+1/4+3/4} - \frac{1}{s^2+s+1/4+3/4}\right]$$

$$= \frac{1}{2}\left[\frac{1}{(s-1/2)^2+(\sqrt{3}/2)^2} - \frac{1}{(s+1/2)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right]$$

$$L^{-1}\frac{s}{s^4+s^2+1} = \frac{1}{2}\left\{L^{-1}\frac{1}{(s-1/2)^2+(\sqrt{3}/2)^2} - L^{-1}\frac{1}{(s+1/2)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\}$$

$$= 1/2\left\{e^{1/2t}L^{-1}\frac{1}{s^2+(\sqrt{3}/2)^2} - e^{-1/2t}\cdot L^{-1}\frac{1}{(s+1/2)^2+(\sqrt{3}/2)^2}\right\}$$

$$= 1/2\, e^{t/2}\frac{1}{\sqrt{3}/2}\sin\sqrt{3}/2 - e^{-1/2t}\sin\sqrt{3}/2\, t \times \sqrt{3}/2$$

$$= \frac{2}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}\left(\frac{e^{t/2}-e^{-t/2}}{2}\right) = \frac{2}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\,\frac{\sinh t}{2}$$

o) $L^{-1}\frac{s}{s^4+4a^4}$

$$s^4+4a^4 = s^4+4a^2s^2+4a^4-4a^2s^2$$

$$= (s^2+2a^2)^2-(2as)^2$$

$$= [s^2+2a^2+2as][s^2+2a^2-2as]$$

$$\frac{s}{s^4+4a^4} = \frac{s}{[s^2+2a^2+2as][s^2+2a^2-2as]}$$

$$= \frac{1}{4a}\left[\frac{1}{s^2-2as+2a^2} - \frac{1}{s^2+2as+2a^2}\right]$$

$$= \frac{1}{4}\left[\frac{1}{[s-a]^2+a^2} - \frac{1}{[s+a]^2+a^2}\right]$$

$$L^{-1}\frac{s}{s^4+4a^4} =$$

$$= \frac{1}{4a}\left(L^{-1}\frac{1}{(s-a)^2+a^2} - L^{-1}\frac{1}{[s+a]^2+a^2}\right)$$

$$= \frac{1}{4a}\left(e^{at}L^{-1}\frac{1}{s^2+a^2} - e^{-at}L^{-1}\frac{1}{s^2+a}\right)$$

$$= \frac{1}{4a}\left(\frac{e^{at}}{a}\sin at - \frac{e^{-at}}{a}\sin at\right)$$

$$= \frac{1}{2a^2}\sin at\cdot\left(\frac{e^{at}-e^{-at}}{2}\right)$$

$$= \frac{1}{2a^2}\sin at\cdot\sinh at$$

$$\frac{1}{(s^2-2as+2a^2)} - \frac{1}{s^2+2as+2a^2}$$

$$= \frac{s^2+2as+2a^2-s^2+2as-2a^2}{(s^2-2as+2a^2)(s^2+2as+2a^2)}$$

$$\frac{1}{4a}\left[\frac{1}{s^2-2as+2a^2} - \frac{1}{s^2+2as+2a^2}\right]$$

$$= \frac{s}{(s^2-2as+2a^2)}$$

$$(s^2+2as+2a^2)$$

p) $\mathcal{L}^{-1} \dfrac{2s^2 - 6s + 5}{s^3 - 6s^2 + 11s - 6}$

$\dfrac{2s^2 - 6s + 5}{s^3 + 6s^2 + 11s - 6} = \dfrac{2s^2 - 6s + 5}{[s-1][s-2][s-3]}$

Let $\dfrac{2s^2 - 6s + 5}{s^3 - 6s^2 + 11s - 6} = \dfrac{A}{s-1} + \dfrac{B}{s-2} + \dfrac{C}{s-3}$

$= \dfrac{A[s-2][s-3] + B[s-1][s-3] + C[s-1][s-2]}{(s-1)(s-2)(s-3)}$

$2s^2 + 6s + 5 = A(s-2)(s-3) + B(s-1)(s-3) + C(s-1)(s-2)$

$s = 1$

$2 - 6 + 5 = A(1-2)(1-3) \quad 2A = 1 \quad \underline{A = 1/2}$

$s = 2$

$2 \times 4 - 6 \times 2 + 5 = 0 + B(2-1)(2-3) + 0$

$1 = -B \quad \underline{B = -1}$

$s = 3$

$2 \times 3^2 - 6 \times 3 + 5 = 0 + 0 + C(3-1)(3-2)$

$5 = 2C$

$C = 5/2$

$\dfrac{2s^3 - 6s + 5}{s^3 - 6s^2 + 11s - 6} = \dfrac{1/2}{s-1} - \dfrac{1}{s-2} + \dfrac{5/2}{s-3}$

$\mathcal{L}^{-1} \dfrac{2s^3 - 6s + 5}{s^3 - 6s^2 + 11s - 6} = \dfrac{1}{2}\mathcal{L}^{-1}\left(\dfrac{1}{s-1}\right) - \mathcal{L}^{-1}\dfrac{1}{s-2} + \dfrac{5}{2}\mathcal{L}^{-1}\dfrac{1}{s-3}$

$= \dfrac{1}{2} e^{t} - e^{2t} + \dfrac{5}{2} e^{3t}$

q) $\mathcal{L}^{-1} \dfrac{1}{s^3 - a^3}$

$= \dfrac{1}{(s-a)(s^2 + as + a^2)}$

Let $\dfrac{1}{s^3 - a^3} = \dfrac{A}{s-a} + \dfrac{Bs + C}{s^2 + as + a^2}$

$= \dfrac{A(s^2 + as + a^2) + (Bs + C)(s-a)}{(s-a)(s^2 + as + a^2)}$

$1 = A(s^2 + as + a^2) + (Bs + C)(s-a)$

Put $s = a$

$1 = A(a^2 + a^2 + a^2)$

$A = \dfrac{1}{3a^2}$

Equaler coe $s^2$

$0 = A + B$

$B = -A$

$= -\frac{1}{3A^2}$

Put $s = 0$

$1 = Aa^2 - Ca$

$= a^2 \times \frac{1}{3a^2} - Ca$

$1 = \frac{1}{3} - Ca$

$C = \frac{-2}{3a}$

$$\frac{1}{s^3 - a^3} = \frac{1/3a^2}{s-a} + \frac{\left(-\frac{1}{3a^2} s - \frac{2}{3a}\right)}{s^2 + as + a^2}$$

$$= \frac{1/3a^2}{s-a} - \frac{1}{3a^2}\frac{s}{s^2 + as + a^2} - \frac{2}{3a}\frac{1}{s^2 + as + a^2}$$

$$= \frac{\frac{1}{3a^2}}{s-a} - \frac{1}{3a^2}\frac{s}{(s+\frac{1}{2})^2 + \left(\frac{\sqrt{3}}{2a}\right)^2} - \frac{2}{3a}\frac{1}{(s+\frac{1}{2})^2 + \left(\frac{\sqrt{3}a}{2}\right)^2}$$

$$= \frac{1/3a^2}{s-a} - \frac{1}{3a^2}\frac{(s + \frac{1}{2} - \frac{1}{2})}{(s+\frac{1}{2})^2 + \left(\frac{\sqrt{3}}{2}\right)^2} - \frac{2}{3a}\frac{1}{(s+\frac{1}{2})^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$L^{-1}\frac{1}{s^3 - a^3} = \frac{1}{3a^2} L^{-1}\frac{1}{s-a} - \frac{1}{3a^2} L^{-1}\frac{(s+\frac{1}{2})}{(s+\frac{1}{2})^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$+ \frac{1}{6a^2} L^{-1}\frac{1}{(s+\frac{1}{2})^2 + \left(\frac{\sqrt{3}}{2}\right)^2} - \frac{2}{3a} L^{-1}\frac{1}{(s+\frac{1}{2})^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \frac{1}{3a^2} e^{at} - \frac{1}{3a^2} e^{-t/2} L^{-1}\frac{s}{s^2 + \left(\frac{\sqrt{3}}{2}\right)^2} + \frac{1}{6a^2} e^{t/2} L^{-1}\frac{1}{s^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$- \frac{2}{3a} e^{-t/2} \cdot L^{-1}\frac{1}{s^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \frac{1}{3a^2} e^{at} - \frac{1}{3a^2} e^{-t/2} \cos\frac{\sqrt{3}}{2}t + \frac{1}{6a^2} e^{-t/2} \cdot \frac{1}{\sqrt{3}} \sin\frac{\sqrt{3}t}{\cancel{2}}$$

$$- \frac{2}{3a} \times \frac{1}{\sqrt{3}/2} e^{-\frac{1}{2}t} \cdot \sin\frac{\sqrt{3}}{2}t$$

$$\mathcal{L}^{-1} \frac{s}{(s^2+a^2)}$$

$$= \cos at * \frac{1}{a} \sin at$$

$$= \frac{1}{a} \int_0^t \cos au \cdot \sin a(t-u)\, du$$

$$= \frac{1}{2a} \int_0^t 2\sin a(t-u) \cos au \, du$$

$$= \frac{1}{2a} \int_0^t \left[ \sin(a(t-u) + au) + \sin(a(t-u) - au) \right] \cdot du$$

$$= \frac{1}{2a} \int_0^t (\sin at \cdot \sin(at - 2au)\, du$$

$$= \frac{1}{2a} \left[ \sin at \int_0^t du + \int_0^t \sin(at - 2au)\, du \right]$$

$$= \frac{1}{2a} \left[ \sin at \; u \right]_0^t - \frac{1}{-2a} \left\{ \cos(at - 2au) \right\}_0^t$$

$$= \frac{1}{2a} \left( t \sin at + \frac{1}{2a} (\cos(-at) - \cos at \right)$$

$$= \frac{1}{2a} \left( t \sin at + \frac{1}{2a} (\cos at - \cos at \right)$$

$$= \frac{1}{2a} t \sin at$$

$$\mathcal{L}^{-1} \frac{s}{(s^2+a^2)^2}$$

$$= \mathcal{L}^{-1} \frac{s}{s^2+a^2} \cdot \frac{1}{s^2+a^2}$$

$$\mathcal{L}^{-1} \frac{s}{s^2+a^2} = \cos at$$

$$\mathcal{L}^{-1} \frac{1}{s^2+a^2} = \frac{1}{a} \sin at$$

$$\rightarrow \mathcal{L}^{-1} \frac{1}{(s^2+a^2)^2}$$

$$= \frac{1}{a} \sin at * \frac{1}{a} \sin at$$

$$= \frac{1}{a^2} \int_0^t \sin au \cdot \sin a(t-u) \cdot du$$

$$= \frac{1}{2a^2} \int_0^t 2\sin a(t-u) \cdot \sin au \, du$$

$$= \frac{1}{2a^2} \int_0^t \left[ (\cos a(t-u) - au) - \cos(a(t-u) + au \right] du$$

$$= \frac{1}{2a^2} \int_0^t \left[ (\cos(at - 2au)\, du - \cos at \int 1\, du \right]$$

$$= \frac{1}{2a^2} \left[ \frac{1}{-2a} \sin(at - 2au) \right\}_0^t - \cos at \, u \Big\}_0^t$$

$$= \frac{1}{2a^2} \left[ -\frac{1}{2a} (\sin(-at) - \sin at) - (t - 0) \cos at \right]$$

$$= \frac{1}{2a^2} \left[ \frac{1}{2a} \times 2 \sin at - t \cos at \right]$$

$$= \frac{1}{2a^3} \left[ \sin at - at \cos at \right]$$

$$\mathcal{L}^{-1}\frac{s^2}{[s^2+a^2][s^2+b^2]}$$ where $a \neq b$

$$= \mathcal{L}^{-1}\frac{s}{(s^2+a^2)}\cdot\frac{s}{(s^2+b^2)}$$

$$= \mathcal{L}^{-1} \cos at * \cos bt$$

$$= \int_0^t \cos au \cdot \cos b(t-u)\, du$$

$$= \frac{1}{2}\int_0^t 2\cos b(t-u)\cdot \cos au\, du$$

$$= \frac{1}{2}\int_0^t \left[\{\cos b(t-u)+au\} + \cos(b(t-u)-au)\right] du$$

$$= \frac{1}{2}\int_0^t \left[\cos(bt-(b-a)u) + \cos(bt-(b+a)u)\right] du$$

$$= \frac{1}{2}\left[\frac{1}{-(b-a)}\sin(bt-(b-a)u + \frac{1}{-(b+a)}\sin(bt-(b+a)u)\right]_0^t$$

$$= \frac{1}{2}\left[\frac{-1}{b-a}(\sin at - \sin bt) - \frac{1}{b+a}(-\sin at - \sin bt)\right]$$

$$= \frac{1}{2}\left[\frac{1}{b-a}(\sin bt - \sin at) + \frac{1}{b+a}(\sin at + \sin bt)\right]$$

$$= \frac{1}{2}\left[\sin bt\left[\frac{1}{b-a}+\frac{1}{b+a}\right] + \sin at\left[\frac{1}{b+a}-\frac{1}{b-a}\right]\right]$$

$$= \frac{1}{2}\left[\frac{2b}{b^2-a^2}\sin bt - \frac{2a}{b^2-a^2}\sin at\right]$$

$$= \frac{b}{b^2-a^2}\sin bt - \frac{a}{b^2-a^2}\sin at$$

$$\mathcal{L}^{-1}\frac{1}{s(s^2+4)}$$

$$\mathcal{L}^{-1}\frac{1}{s} = 1$$

$$\mathcal{L}^{-1}\frac{1}{s^2+4} = \frac{1}{2}\sin 2t$$

$$= \mathcal{L}^{-1}\frac{1}{s}\cdot\frac{1}{s^2+4}$$

$$= 1 * \frac{1}{2}\sin 2t$$

$$= \int_0^t \frac{1}{2}\sin 2(t-u)\, du$$

$$= \frac{1}{2}\times\frac{-1}{-2}\left[\cos(2t-2u)\right]_0^t$$

$$= \frac{1}{4}\left[\cos 0 - \cos 2t\right]$$

$$= \frac{1}{4}(1-\cos 2t)$$

$$\mathcal{L}^{-1}\frac{1}{s^2(s^2-a^2)}$$

$$= \mathcal{L}^{-1}\frac{1}{s^2}\cdot\frac{1}{s^2-a^2}$$

$$= t * \frac{1}{a}\sinh at$$

$$= \int_0^t u\cdot\frac{1}{a}\sinh a(t-u)\, du$$

$$= \frac{1}{a}\int_0^t \left[u\cdot\frac{1}{-a}\cosh a(t-u)\right]_0^t - \int_0^t -\cosh a(t-u)\, du$$

$$= \frac{1}{a}\left(\frac{-1}{a}(t\cosh 0 - 0) + \frac{1}{-a}\sinh a(t-u)\right)_0^t$$

$$= \frac{1}{a}\left[-\frac{t}{a}\cdot 1 - \frac{1}{a}\left(\sinh a(t-t) - \sinh at\right)\right]$$

$$= \frac{1}{a^2}\left(-t - (0 - \sinh at)\right]$$

$$= \frac{1}{a^2}\left[\sinh at - t\right]$$

. $L^{-1}\dfrac{s}{(s^2+a^2)(s^2+b^2)}$

$$= L^{-1}\frac{s}{s^2+a^2}\cdot\frac{1}{s^2+b^2}$$

$$= \cos at * \frac{1}{b}\sin bt$$

$$= \frac{1}{b}\int_0^t \cos au \cdot \sin b(t-u)\,du$$

$$= \frac{1}{2b}\int_0^t 2\cos au \sin b(t-u)\,du$$

$$= \frac{1}{2b}\int_0^t \left(\sin(au+b(t-u)) - \sin(au-b(t-u))\right)du$$

$$= \frac{1}{2b}\left[\int_0^t \sin((a-b)u+bt) - \sin((a+b)u - bt)\right]du$$

$$= \frac{1}{2b}\left[\frac{-1}{a-b}\cos((a-b)u+bt) + \frac{1}{a+b}\cos((a+b)u - bt\right]_0^t$$

$$= \frac{1}{2b}\left[\frac{-1}{a-b}(\cos at - \cos bt) + \frac{1}{a+b}(\cos at - \cos(-bt))\right)$$

$$= \frac{1}{2b}\left[\frac{-1}{a-b}(\cos at - \cos bt) + \frac{1}{a+b}(\cos at - \cos bt)\right)$$

$$= \frac{1}{2b}\cos(at-bt)\left[\frac{1}{a+b} - \frac{1}{a-b}\right]$$

$$= \cos(at-bt)\cdot\frac{-1}{(a^2-b^2)}$$

$$= \frac{1}{b^2-a^2}(\cos at - \cos bt)$$

. $L^{-1}\dfrac{s}{(s^2+4)(s^2+9)}$

$$= L^{-1}\frac{1}{s^2+4}\cdot\frac{s}{s^2+9}$$

$$= \frac{1}{2}\sin 2t * \cos 3t$$

$$= \frac{1}{2}\int_0^t \sin 2u \cdot \cos 3(t-u)\,du$$

$$= \frac{1}{4}\int_0^t 2\sin 2u \cdot \cos 3(t-u)\,du$$

$= \frac{1}{4}\int_0^t \left[ \sin(2u+3(t-u)) + \sin(2u-3(t-u)) \right] du$

$= \frac{1}{4}\int_0^t \left[ \sin(3t-u) + \sin(5u-3t) \right] \cdot du$

$= \frac{1}{4}\left[ \frac{-1}{-1}\cos(3t-u) - \frac{1}{5}\cos 2t - \cos(-3t) \right]$

$= \frac{1}{4}\left(\left(\cos 2t - \cos 3t\right) - \frac{1}{5}\left(\cos 2t - \cos 3t\right)\right)$

$= \frac{1}{4} \cos 2t - \cos 3t \left(1 - \frac{1}{5}\right)$

$= \frac{1}{4} \times \frac{4}{5}\left(\cos 2t - \cos 3t\right)$

$= \frac{1}{5}\left(\cos 2t - \cos 3t\right)$

APPLICATION OF LAPLACE TRANSFORM BY TO THE DIFFERENTIAL EQUATION

1. Solve $y'' + 5y' + 6y = 5e^{2t}$

Given $y(0)=2$, $y'(0)=1$

$L(y'') + 5L(y') + 6L(y) = 5L(e^{2t})$

$s^2 L(y) - s\,y(0) - y'(0) + 5(sL(y) - y(0)) + 6L(y)$

$= \frac{5}{s-2}$

$= s^2 L(y) - 2s - 1 + 5[sL(y) - 2] + 6L(y) = \frac{5}{s-2}$

$= L(y)(s^2+5s+6) - 2s - 1 - 10 = \frac{5}{s-2}$

$(s+3)(s+2)L(y) = 2s + 11 + \frac{5}{s-2}$

$= \frac{2s^2 - 4s + 11s - 22 + 5}{s-2}$

$= \frac{2s^2 + 7s - 1}{s-2}$

$L(y) = \frac{2s^2 + 7s - 17}{(s-2)(s+2)(s+3)}$

Let $\frac{2s^2+7s-1}{(s-2)(s+2)(s+3)} = \frac{A}{s-2} + \frac{B}{s+2} + \frac{C}{s+3}$

$= \frac{A(s+2)(s+3) + B(s-2)(s+3) + C(s-2)(s+2)}{(s-2)(s+2)(s+3)}$

$2s^2 + 7s - 17 = A(s+2)(s+3) + B(s-2)(s+3) + C(s-2)(s+2)$

Put $s=2$ ; $5 = 20A \Rightarrow A = 1/4$

Put $s=-2$; $-23 = -4B \Rightarrow B = 23/4$

Put $s=-3$ ; $-26 = 5C \Rightarrow C = -4$

$L^{-1} \frac{2s^2+7s-17}{(s-2)(s+2)(s+3)} = \frac{1}{4} L^{-1}\frac{1}{s-2} + \frac{23}{4} L^{-1}\frac{1}{s+2} - 4L^{-1}\frac{1}{s+3}$

$= \frac{1}{4} e^{2t} + \frac{23}{4} e^{-2t} - 4 e^{-3t}$

Q Solve $y''' + 2y'' - y' - 2y = 0$

Given $y(0) = y'(0) = 0$ ; $y''(0) = 6$

$L(y''') + 2L(y'') - L(y') - 2L(y) = 0$

$s^3 L(y) - s^2(y(0)) - s y'(0) - y''(0)$

$+ 2(s^2 L(y) - s y(0) - y'(0)) - (s L(y) - y(0))$

$- 2L(y) = 0$

$s^3 L(y) - 0 - 0 - 6 + 2(s^2 L(y) - 0 - 0) - (s L(y) - 0)$

$- 2(y) = 0$

$L(y)(s^3 + 2s^2 - s - 2) = 6$

$L(y)(s-1)(s+1)(s+2) = 6$

$L(y) = \frac{6}{(s-1)(s+1)(s+2)}$

Lit. $\frac{6}{(s-1)(s+1)(s+2)} = \frac{A}{s-1} + \frac{B}{s+1} + \frac{C}{s+2}$

$6 = A(s+1)(s+2) + B(s-1)(s+2) + C(s-1)(s+1)$

Put $s = 1$ ; $6 = 6A \Rightarrow A = 1$

Put $s = -2$ ; $6 = 3C$ $C = 2$

Put $s = -1$ ; $6 = 2B$ $B = 3$

$L(y) = \frac{1}{s-1} + \frac{3}{s+1} + \frac{2}{s+2}$

$y = L^{-1}\frac{1}{s-1} + 3L^{-1}\frac{1}{s+1} + 2L^{-1}\frac{1}{s+2}$

$y = e^{t} + 3e^{-t} + 2e^{-2t}$

Q. Solve $(D^2 + n^2)x = \sin(nt + \alpha)$

Given ~~x(0)~~ $x = Dx = 0$ when $t = 0$

$d^2 x + n^2 x = \sin nt \cos\alpha + \cos nt \sin\alpha$

$x'' + n^2 x = \sin nt \cos\alpha + \cos nt \sin\alpha$

$L(x'') + n^2 L(x) = \cos\alpha L(\sin nt) + \sin\alpha L(\cos nt)$

$s^2 L(x) - s x(0) - x'(0) + n^2 L(x) = \cos\alpha \frac{n}{s^2+n^2} + \sin\alpha \frac{s}{s^2+n^2}$

$s^2 L(x) - 0 - 0 + n^2 L(x) = \cos\alpha \frac{n}{s^2+n^2} + \sin\alpha \frac{s}{s^2+n^2}$

$L(x)(s^2 + n^2) = \cos\alpha \frac{n}{(s^2+n^2)^2} + \sin\alpha \frac{s}{(s^2+n^2)^2}$

$x = n\cos\alpha L^{-1}\frac{1}{(s^2+n^2)^2} + \sin\alpha L^{-1}\frac{s}{(s^2+n^2)^2}$

$$x = n\alpha \cdot \frac{1}{2n^2}\left[\sin nt - nt\cos nt\right) + \sin\alpha \frac{1}{2n}\, t\sin nt\Big]$$

Q. $\frac{d^2y}{dt^2} + n^2y = a\cos(nt+\alpha)$

given $y(0) = y'(0) = 0$

$$\frac{d^2y}{dt^2} + n^2y = a\left[\cos\alpha\cos nt - \sin nt\sin\alpha\right)$$

$$y'' + n^2y = a\cos nt\cos\alpha - a\sin nt\sin\alpha$$

$$L(y'') + n^2L(y) = a\cos\alpha\, L(\sin nt) - a\sin\alpha\, L(\sin nt)$$

$$s^2L(y) - s\,y(0) - y'(0) + n^2L(y) = a\cos\alpha \frac{n}{n^2+s^2} - a\sin\alpha \frac{n}{s^2+n^2}$$

$$s^2L(y) - 0 - 0 + n^2L(y) = a\cos\alpha \frac{s}{s^2+n^2} - a\sin\alpha \frac{n}{s^2+n^2}$$

$$L(y)(s^2+n^2) = a\cos\alpha \frac{s}{s^2+n^2} + \sin\alpha \frac{n}{s^2+n^2}$$

$$L(y) = a\left(\cos\alpha \frac{s}{(s^2+n^2)^2} + \sin\alpha \frac{n}{(s^2+n^2)^2}\right)$$

$$x = a\cos\alpha \frac{s}{(s^2+n^2)}$$

$$x = a\cos\alpha\, L^{-1}\frac{s}{(s^2+n^2)} + na\sin\alpha\, L^{-1}\frac{1}{(s^2+n^2)^2}$$

$$x = a\cos\alpha \frac{1}{2n}\, t\sin nt + na\sin\alpha \frac{1}{2n^3}\left(\sin nt - nt\cos nt\right]$$

SIMULTANEOUS EQUATION

- $\frac{dx}{dt} + y = \sin t$

- $x + \frac{dy}{dt} = \cos t$

Given $x = 2, y = 0$ for $t = 0$

$$x' + y = \sin t$$

$$x + y' = \cos t$$

$$L(x') + L(y) = L(\sin t)$$

$$L(x) + L(y') = L(\cos t)$$

$$s\,L(x) - x(0) + L(y) = \frac{1}{s^2+1}$$

$$L(x) + s\,L(y) - y(0) = \frac{s}{s^2+1}$$

Let $L(x) = \bar{x}$, $L(y) = \bar{y}$

$$s\bar{x} - 2 + \bar{y} = \frac{1}{s^2+1}$$

$$\bar{x} + s\bar{y} - 0 = \frac{s}{s^2+1}$$

$$s\bar{x} + \bar{y} = 2 + \frac{1}{s^2+1}$$

$$\bar{x} + s\bar{y} = \frac{s}{s^2+1}$$

$$s^2\bar{x} + s\bar{y} = 2s + \frac{s}{s^2+1}$$

$$\bar{x} + s\bar{y} = \frac{s}{s^2+1}$$

$$(s^2-1)\bar{x} = 2s$$

$$\bar{x} = \frac{2s}{s^2-1}$$

$$L(x) = 2\frac{s}{s^2-1}$$

$$x = 2L^{-1}\frac{s}{s^2-1}$$

$$= 2\cosh t$$

$$\frac{dx}{dt} = 2\sinh t$$

$$y = \sin t - \frac{dx}{dt}$$

$$= \sin t - 2\sinh t$$

Solve $\frac{dx}{dt} - y = e^t$

$$x + \frac{dy}{dt} = \sin t.$$

given $x(0)=1$, $y(0)=0$

$$x' - y = e^t$$

$$x + y' = \sin t$$

$$L(x') - L(y) = L(e^t)$$

$$L(x) + L(y') = L(\sin t)$$

$$sL(x) - x(0) - L(y) = \frac{1}{s-1}$$

$$L(x) + s(L(y)) - y(0) = \frac{1}{s^2+1}$$

Let $L(x) = \bar{x}$ $L(y) = \bar{y}$

$$s\bar{x} - 1 - \bar{y} = \frac{1}{s-1}$$

$$\bar{x} + s\bar{y} - 0 = \frac{1}{s^2+1}$$

$$s\bar{x} - \bar{y} = 1 + \frac{1}{s-1} = \frac{s}{s-1}$$

$$\bar{x} + s\bar{y} = \frac{1}{s^2+1}$$

$$(s^2+1)\bar{x} = \frac{s^2}{s-1} + \frac{1}{s^2+1}$$

$$\bar{x} = \frac{s^2}{(s-1)(s^2+1)} + \frac{1}{(s^2+1)^2}$$

$$L(x) = \frac{s^2}{(s-1)(s^2+1)} + \frac{1}{(s^2+1)^2}$$

Let $\frac{s^2}{(s-1)(s^2+1)}$

$$= \frac{A}{s-1} + \frac{Bs+C}{s^2+1}$$

$$= \frac{A(s^2+1) + (Bs+C)(s-1)}{(s-1)(s^2+1)}$$

$$s^2 = A(s^2+1) + (Bs+C)(s-1)$$

Put $s=1$

$1 = 2A \Rightarrow A = \frac{1}{2}$

$s=0$

$0 = A - C \Rightarrow C = \frac{1}{2}$

Equ. this with $s^2$

$1 = A + B = \frac{1}{2} + B$

$B = \frac{1}{2}$

$$L(x) = \frac{\frac{1}{2}}{s-1} + \frac{\frac{1}{2}s + \frac{1}{2}}{s^2+1} + \frac{1}{(s^2+1)^2}$$

$$x = \tfrac{1}{2}L^{-1}\frac{1}{s-1} + \tfrac{1}{2}L^{-1}\frac{s}{s^2+1} + \tfrac{1}{2}L^{-1}\frac{1}{s^2+1} + L^{-1}\frac{1}{(s^2+1)^2}$$

$$= \tfrac{1}{2}e^t + \tfrac{1}{2}\cos t + \tfrac{1}{2}\sin t + \frac{1}{2s^3}(\sin t - t\cos t)$$

$$= \tfrac{1}{2}e^t + \tfrac{1}{2}\cos t + \sin t - \tfrac{1}{2}t\cos t$$

$$\frac{dx}{dt} = \tfrac{1}{2}e^t - \tfrac{1}{2}\sin t + \cos t - \tfrac{1}{2}(\cancel{tt}\sin t \; t\cos t)$$

$$= \tfrac{1}{2}e^t - \tfrac{1}{2}\sin t + \tfrac{1}{2}\cos t - \tfrac{1}{2}t\sin t - \tfrac{1}{2}\cos t$$

$$- \tfrac{1}{2}e^t - \tfrac{1}{2}\sin t + \tfrac{1}{2}\cos t - \tfrac{1}{2}t\sin t$$

$$y = \frac{dy}{dt} - e^t$$

$$= -\tfrac{1}{2}e^t - \tfrac{1}{2}\sin t + \tfrac{1}{2}\cos t + \tfrac{1}{2}t\sin t$$

$0.\ \cancel{x' - y = e^{t}}$ $\quad x' + y = \sin t$

$\cancel{x' + y} = -$ $\quad -x + y' = e^t$

Given $x(0) = 0,\ y(0) = 1$

$L(x') + L(y) = L(\sin t)$

$-L(x) + L(y') = L(e^t)$

$$sL(x) + L(y) = \frac{1}{s^2+1}$$

$$-L(x) + sL(y) = \frac{1}{s-1}$$

Let $L(x) = \bar{x}$ $\quad L(y) = \bar{y}$

$$s\bar{x} + \bar{y} = \frac{1}{s^2+1}$$

$$-\bar{x} + \bar{y}s \underset{-y(0)}{} = \frac{1}{s-1}$$

$$s^2\bar{x} + s\bar{y} = \frac{s}{s^2+1}$$

$$-s\bar{x} + \bar{y}s^2 = \frac{s}{s-1} + s$$

$$(s^2+1)\bar{y} = \frac{s}{(s^2+1)^2} - \frac{s^2}{s-1(s^2+1)}$$

$$\bar{y} = \frac{s^2}{(s^2+1)^2(s-1)}$$

Let $\frac{s^2}{(s-1)(s^2+1)} = \frac{1/2}{s-1} + \frac{\frac{1}{2}s + \frac{1}{2}}{s^2+1}$

$$\bar{y} = \frac{1}{(s^2+1)^2} + \frac{1/2}{s-1} + \frac{\frac{1}{2}s}{s^2+1} + \frac{1/2}{s^2+1}$$

$$y = L^{-1}\frac{1}{(s^2+1)^2} + \tfrac{1}{2}L^{-1}\frac{1}{s-1} + \tfrac{1}{2}L^{-1}\frac{s}{s^2+1} + \tfrac{1}{2}L^{-1}\frac{1}{s^2+1}$$

$y' = \frac{1}{2(1)^3}\left[\sin t - t\cos t\right] + \frac{1}{2}e^t + \frac{1}{2}\cos t + \frac{1}{2}\sin t$

$y = \frac{1}{2}\sin t - \frac{1}{2}t\cos t + \frac{1}{2}e^t + \frac{1}{2}\cos t + \frac{1}{2}\sin t$

$y = \frac{1}{2}e^t + \sin t + \frac{1}{2}\cos t - \frac{1}{2}t\cos t$

$\frac{dy}{dt} = \frac{1}{2}e^t + \cos t - \frac{1}{2}\sin t - \frac{1}{2}(t\cancel{x} - \sin t + \cos t)$

$= \frac{1}{2}e^t + \cos t - \frac{1}{2}\sin t + \frac{1}{2}t\sin t - \frac{1}{2}\cos t$

$= \frac{1}{2}e^t + \frac{1}{2}\cos t - \frac{1}{2}\sin t + \frac{1}{2}t\sin t$

$x = y' - e^t$

$= -\frac{1}{2}e^t + \frac{1}{2}\cos t - \frac{1}{2}\sin t + \frac{1}{2}t\sin t$

• Solve $\cancel{\frac{d^2x}{} + 3}$ $D^2x + 3x - 2y = 0$

$D^2x + D^2y - 3x + 5y = 0$

Given $x(0) = y(0) = 0$ and $x'(0) = 3$ $y'(0) = 2$

$x'' + 3x - 2y = 0$

$x'' - 3x + y'' + 5y = 0$

$L(x'') + 3L(x) + 2L(y) = 0$

$L(x'') - 3L(x) + L(y'') + 5L(y) = 0$

$s^2L(x) - sx(0) - x'(0) + 3x - 2L(y) = 0$

$s^2L(x) - sx(0) - x'(0) - 3L(x) + s^2L(y) - sy(0)$

$- y'(0) + 5L(y) = 0$

Let $L(x) = \bar{x}$, $L(y) = \bar{y}$

$s^2\bar{x} - 3 + 3\bar{x} - 2\bar{y} = 0$

$s^2x - 3 - 3\bar{x} + s^2\bar{y} - 2 + 5\bar{y} = 0$

$(s^2+3)\bar{x} - 2\bar{y} = 3$

$(s^2-3)\bar{x} + (s^2+5)\bar{y} = 5$

$(s^2+3)(s^2+5)\bar{x} - 2(s^2+5)\bar{y} = 3(s^2+5)$

$2(s^2-3)\bar{x} + 2(s^2+5)\bar{y} = 10$

$(s^2+3)(s^2+5)\bar{x} + 2(s^2-3)\bar{x} = 3(s^2+5) + 10$

$(s^4 + 8s^2 + 15 + 2s^2 - 6)$ $3s^2 + 25$

$s^4 + 10s^2 + 9 = 3s^2 + 25$

$(s^2+9)(s^2+1)\bar{x} = 3s^2 + 25$

$\bar{x} = \frac{3s^2+25}{(s^2+9)(s^2+1)}$

Let $\dfrac{3s^2+2s}{(s^2+9)(s^2+1)}$

$= \dfrac{As+B}{s^2+9} + \dfrac{Cs+D}{s^2+1}$

$= \dfrac{(As+B)(s^2+1) + (Cs+D)(s^2+9)}{(s^2+9)(s^2+1)}$

$3s^2+2s = (As+B)(s^2+1) + (Cs+D)(s^2+9)$

Equaling with $s^2$

$3 = B+D$

Equ. $s^3$

$0 = A+C$

Equ. $s$

$0 = A+9C$

Put $s=0$

$2s = B+9D$

$B+9D = 2s$

$B+D = 3$

$D = 11/4$

$9C + A = 0$

$C + A = 0$

$8C = 0$

$C = 0$

$A = -C = 0$

$B+D = 3$

$B = 1/4$

$L(x) = \dfrac{\frac{1}{4}}{s^2+9} + \dfrac{11/4}{s^2+1}$

$x = 1/4\, L^{-1} \dfrac{1}{s^2+9} + 11/4\, L^{-1} \dfrac{1}{s^2+1}$

$= 1/4 \times \frac{1}{3} \sin 3t + 11/4 \sin t$

$= 1/12 \sin 3t + 11/4 \sin t$

$\dfrac{dx}{dt} = \dfrac{1}{12} \times 3 \cos 3t + 11/4 \cos t$

$= 1/4 \cos 3t + 11/4 \cos t$

$\dfrac{d^2x}{dt^2} = -3/4 \sin 3t - 11/4 \sin t$

$2y = D^2x + 3x$

$= -3/4 \sin 3t - 11/4 \sin t + 3/12 \sin 3t + \frac{33}{4} \sin t$

$= -6/12 \sin 3t + \frac{22}{4} \sin t$

$2y = -1/2 \sin 3t + 11/2 \sin t$

$y = -1/4 \sin 3t + 11/4 \sin t$

APPLICATION OF LAPLACE TRANSFORM OF INTEGRAL ~~DIFFERENTIAL~~ EQUATION

Solve $\dfrac{dy}{dt} + 3y + 2\int_0^t y\, dt = t$

given $y(0) = 1$

$$L(y') + 3L(y) + 2L\int_0^t y\,dt = L(t)$$

$$sL(y) - y(0) + 3L(y) + 2\frac{1}{s}L(y) = 1/s^2$$

$$sL(y) - 1 + 3L(y) + \frac{2}{s}L(y) = 1/s^2$$

$$L(y)\left(s+3+2/s\right) = 1 + 1/s^2$$

$$L(y)\left[\frac{s^2+3s+2}{s}\right] = \frac{s^2+1}{s^2}$$

$$L(y)(s+2)(s+1) = \frac{s^2+1}{s}$$

$$L(y) = \frac{s^2+1}{s(s+1)(s+2)}$$

Let $\dfrac{s^2+1}{s(s+1)(s+2)} = \dfrac{A}{s} + \dfrac{B}{s+1} + \dfrac{C}{s+2}$

$$s^2+1 = A(s+1)(s+2) + B(s)(s+2) + C(s)(s+1)$$

$s=-1 \Rightarrow 2 = -B \Rightarrow B = -2$

$s=-2 \Rightarrow 5 = 2C \Rightarrow C = 5/2$

Put $s=0$ $\quad 1 = 2A \Rightarrow A = 1/2$

$$L(y) = \frac{1/2}{s} - \frac{2}{s+1} + \frac{3/2}{s+2}$$

$$y = 1/2\, L^{-1}\, 1/s - 2L^{-1}\, 1/(s+1) + 5/2\, L^{-1}\, 1/(s+2)$$

$$y = 1/2 - 2e^{-t} + \frac{5}{2}e^{-2t}$$

Q. $\dfrac{dy}{dt} + 4y + 5\displaystyle\int_0^t y\,dt = e^{-t}$

given $y(0) = 0$

$$y' + 4y + 5\int_0^t y\,dy = e^{-t}$$

$$L(y') + 4L(y) + 5L\int_0^t y\,dt = L(e^{-t})$$

$$sL(y) - y(0) + 4L(y) + 5\,\frac{1}{s}\,L(y) = \frac{1}{s+1}$$

$$sL(y) - 0 + 4L(y) + \frac{5}{s}L(y) = \frac{1}{s+1}$$

$$L(y)\left(s+4+\frac{5}{s}\right) = \frac{1}{s+1}$$

$$L(y)\left[\frac{s^2+4s+5}{s}\right] = \frac{1}{s+1}$$

$$L(y) = \frac{s}{(s+1)(s^2+4s+5)}$$

Let $\dfrac{s}{(s+1)(s^2+4s+5)} = \dfrac{A}{s+1} + \dfrac{Bs+C}{s^2+4s+5}$

$s = A(s^2+4s+4) + (s+1)(Bs+C)$

Coeff $s^2$

$0 = A + B$

Eq. coef $s$

$1 = 4A + C + B$

Put $s = 0$

$0 = 4A + C$

$4A + C + B = 1$

$4A + C - A = 1 \Rightarrow 3A + C = 1$

~~$B = 1$~~

~~$A = B = 1$~~

$4A + C = 0 \Rightarrow 2A = -1$

$A = -1/2$

$C = 5/2$

$$L(y) = \frac{1/2}{s+1} + \frac{\frac{1}{2}s + 5/2}{s^2+4s+5}$$

$$= \frac{1/2}{s+1} + \frac{\frac{1}{2}s + 5/2}{(s+2)^2+1}$$

$$= \frac{1/2}{s+1} + \frac{\frac{1}{2}(s+5)}{(s+2)^2+1} = \frac{1/2}{s+1} + \frac{1}{2}\frac{(s+2+3)}{(s+2)^2+1}$$

$$= \frac{1/2}{s+1} + \frac{s+2}{(s+2)^2+1} \times \frac{1}{2} + \frac{3/2}{(s+2)^2+1}$$

$$y = \frac{1}{2} L^{-1}\frac{1}{s+1} + \frac{1}{2} L^{-1}\frac{s+2}{(s+2)^2+1} + \frac{3}{2} L^{-1}\frac{1}{(s+2)^2+1}$$

$$= \frac{1}{2}e^{-t} + \frac{1}{2}e^{-2t} L^{-1}\frac{s}{s^2+1} + \frac{3}{2}e^{-2t} L^{-1}\frac{1}{s^2+1}$$

$$= \frac{1}{2}e^{-t} + \frac{1}{2}e^{-2t}\cos t + \frac{3}{2}e^{-2t}\sin t$$

---

• Solve $y(t) = e^{-t} - 2\int_0^t y(u)\sin(t-u)\,du$.

$= e^{-t} - 2\, y(t) * \sin t$

$L(y) = L(e^{-t}) - 2L(y)\,L(\sin t)$

$$= \frac{1}{s+1} - 2L(y)\frac{1}{s^2+1}$$

$$L(y) + 2L(y)\frac{1}{s^2+1} = \frac{1}{s+1}$$

$$L(y)\left[1 + \frac{2}{s^2+1}\right] = \frac{1}{s+1}$$

$$L(y)\left[\frac{s^2+1+2}{s^2+1}\right] = \frac{1}{s+1}$$

$$L(y) = \frac{s^2+1}{(s^2+3)(s+1)}$$

Let $\frac{s^2+1}{(s^2+3)(s+1)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+3}$

$$= \frac{A(s^2+3) + (Bs+C)(s+1)}{(s^2+3)(s+1)}$$

Put $s = -1$

$2 = 4A \Rightarrow A = 1/2$

Put $s = 0$

$1 = 3A + C \Rightarrow C = -1/2$

Equat. coeff $s^2$

$1 = A + B$

$B = 1/2$

$$\frac{s^2+1}{(s+1)(s^2+3)} = \frac{1/2}{s+1} + \frac{1/2 s - 1/2}{s^2+3}$$

$$L^{-1}\frac{s^2+1}{(s+1)(s^2+3)} = 1/2\, L^{-1}\frac{1}{s+1} + 1/2\, L^{-1}\frac{s}{s^2+3} - 1/2\, L^{-1}\frac{1}{s^2+3}$$

$$\therefore y = 1/2\, e^{-t} + 1/2 \cos\sqrt{3}t - 1/2 \times \frac{1}{\sqrt{3}} \sin\sqrt{3}t$$

- $y(t) = e^{-t} - 2\int_0^t y(u) \cdot \cos(t-u)\, du$

$y(t) = e^{-t} - 2\int_0 y(t) * \cos t$

$$L(y) = L(e^{-t}) - 2L(y(t) * \cos t)$$

$$= \frac{1}{s+1} - 2L\,y(t) \cdot L(\cos t)$$

$$= \frac{1}{s+1} - 2L(y) \cdot \frac{s}{s^2+1}$$

$$L(y) + 2L(y)\frac{s}{s^2+1} = \frac{1}{s+1}$$

$$L(y)\left(1 + \frac{2s}{s^2+1}\right) = \frac{1}{s+1}$$

$$L(y)\left(\frac{s^2+1+2s}{s^2+1}\right) = \frac{1}{s+1}$$

$$L(y)\frac{(s+1)^2}{s^2+1} = \frac{1}{s+1}$$

$$L(y) = \frac{s^2+1}{(s+1)^3}$$

Let $\frac{s^2+1}{(s+1)^3} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{(s+1)^3}$

$s^2+1 = A(s+1)^2\cancel{(s+1)^3} + B(s+1)\cancel{(s+1)^3}$
$+ C\,\cancel{(s+1)}\cancel{(s+1)^2}$

put $s = -1$ $\qquad s = 0$

$2 = C$ $\qquad 1 = A + B + C$

Equat. .. $s^2$

$1 = A$

$1 = 1 + B + 2$

$B = -2$

$$L(y) = \frac{1}{s+1} - \frac{2}{(s+1)^2} + \frac{2}{(s+1)^3}$$

$y = L^{-1}\frac{1}{s+1} - 2L^{-1}\frac{1}{(s+1)^2} + 2L^{-1}\frac{1}{(s+1)^3}$

$y = e^{-t} - 2e^{-t}L^{-1}\frac{1}{s^2} + 2e^{-t}L^{-1}\frac{1}{s^3}$

$= e^{-t} - 2e^{-t}t + 2e^{-t}\frac{t^2}{\lfloor 2}$

$y = e^{-t} - 2e^{-t}t + t^2e^{-t}$

$= e^{-t}(1-2t+t^2)$

$= (t-1)^2e^{-t}.$

EQUATIONS CONTAINING MULTIPLE OF $t$.

Solve $t.y'' + 2y' + ty = \sin t$

given $y(0)=1$

$L(ty'') + 2L(y') + L(ty) = L(\sin t)$

$-1\frac{d}{ds}L(y'') + 2L(y') + (-1)\frac{d}{ds}L(y) = \frac{1}{s^2+1}$

$-\frac{d}{ds}(s^2L(y) - sy(0) - y'(0)\} + 2(sL(y) - y(0))$

$-\frac{d}{ds}L(y) = \frac{1}{s^2+1}$

$-\frac{d}{ds}\{s^2L(y) - s - y'(0)\} + 2\{sL(y) - 1\}$

$-\frac{d}{ds}L(y) = \frac{1}{s^2+1}$

$-\{s^2\frac{d}{ds}L(y) + 2sL(y) - 1 - 0\} + 2sL(y) - 2 - \frac{d}{ds}L(y) = \frac{1}{s^2+1}$

$-s^2\frac{d}{ds}L(y) - 2L(y) + 1 + 2sL(y) = -2$

$-s^2\frac{d}{ds}L(y) - 1 - \frac{d}{ds}L(y) = \frac{1}{s^2+1}$

$+s^2(-1)\frac{d}{ds}L(y) - 1 = -\frac{d}{ds}L(y) = \frac{1}{s^2+1}$

$-\frac{d}{ds}\{L(y)\}(s^2+1) = \frac{1+1}{s^2+1}$

$-\frac{d}{d}L(y) = \frac{1}{s^2+1} + \frac{1}{(s^2+1)^2}$

$L^{-1}\left(-\frac{d}{ds}L(y)\right\} = L^{-1}\frac{1}{s^2+1} + L^{-1}\frac{1}{(s^2+1)^2}$

$= \sin t + \frac{1}{2(1)^3}(\sin t - t\cos t)$

$ty = \sin t + \frac{1}{2}(\sin t - t\cos t)$

$$y = \frac{\sin t}{2} + \frac{1}{\omega t}(\sin \omega t - b\cos t)$$

NOTE:

$$L(t\, L(y)) = (-1)\frac{d}{ds} L(y)$$

$$t\, L(y) = L^{-1}(-1)\frac{d}{ds}(y).$$

BETA FUNCTION ($\beta$)

The definite integral of form $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ is called $\beta$ function. If it exists and finite

PROPERTIES OF $\beta$ function

- $\beta$ function is symmetric. ie $\beta(m,n) = \beta(n,m)$

Proof

Put $x = 1-y$

$dx = -dy$

when $x = 0$,

$0 = 1-y \Rightarrow y = 1$

when $x = 1$,

$1 = 1-y \Rightarrow y = 0$

$$\beta(m,n) = \int_1^0 (1-y)^{m-1} y^{n-1} (-dy)$$

$$= -\int_1^0 y^{n-1}(1-y)^{m-1} dy$$

$$= \int_0^1 y^{n-1}(1-y)^{m-1} dy$$

$$= \int_0^1 x^{n-1}(1-x)^{m-1} dy$$

$$= \beta(n,m)$$

- If $n$ is a positive integer then

$$\beta(m,n) = \frac{\lfloor n-1}{m(m+1)\ldots(m+n-2)\ldots(m+n-1)}$$

If $m$ and $n$ are +ve integers then $\beta(m,n)$

$$= \frac{\lfloor(n-1) \lfloor m-1}{\lfloor m+n-1}$$

$$= \frac{\Gamma n \, \Gamma m}{\Gamma m+n}$$

Proof:

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} \, dx$$

$$= (1-x)^{n-1} \frac{x^m}{m}\Big]_0^1 - \int_0^1 \frac{x^m}{m}(n-1)(1-x)^{n-2}(-1)$$

$$= 0 - 0 + \frac{(n-1)}{m} \int_0^1 x^m (1-x)^{n-1-1} \, dx$$

$$= \frac{(n-1)}{m} \int_0^1 x^{m+1-1}(1-x)^{n-1-1} \, dx$$

$$= \frac{n-1}{m} \beta(m+1, n-1)$$

$$= \frac{n-1}{m} \cdot \frac{n-2}{(m+1)} \beta(m+2, n-2)$$

$$= \frac{(n-1)}{m} \cdot \frac{(n-2)}{m+1} \cdot \frac{n-3}{m+2} \beta(m+3, n-3)$$

$$= \frac{(n-1)(n-2)(n-3) \ldots \{n-(n-1)\}}{m(m+1)(m+2) \ldots (m+n-2)} \beta(m+n-1, 1)$$

$$= \frac{(n-1)(n-2)(n-3) \ldots 1}{m(m+1)(m+2) \ldots (m+n-2)} \beta(n+m-1, 1)$$

$$= \frac{\lfloor n-1}{m(m+1) \ldots (m+n-2)} \beta(m+n-1, 1)$$

$$= \frac{\lfloor n-1}{m(m+1)(m+2) \ldots (m+n-2)} \int_0^1 x^{m+n-2} (1-x)^{1-1} \, dx$$

$$= \frac{\lfloor n-1}{m(m+1) \ldots (m+n-2)} \int_0^1 x^{m+n-2} \, dx$$

$$= \frac{\lfloor n-1}{m(m+1) \ldots (m+n-2)} \frac{x^{m+n-1}}{m+n-1}\Big]_0^1$$

$$= \frac{\lfloor n-1}{m(m+1) \ldots (m+n-2)(m+n-1)}$$

$$\frac{\lfloor n-1}{(m+n-1)(m+n-2)\ldots(m+1)m}$$

when $m$ is also a +ve integer

$$\beta(m,n) = \frac{\lfloor n-1 \, \lfloor m-1}{(m+n-1)(m+n-2)\ldots m(m-1)(m-2)\ldots 2}$$

$$= \frac{\lfloor n-1 \, \lfloor m-1}{\lfloor m+n-1}$$

3. $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta \, d\theta$

Proof: $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

Put $x = \sin^2\theta$

$dx = 2\sin\theta\cos\theta\,d\theta$

when $x = 0$

$\sin^2\theta = 0$

$\theta = 0$

when $x = 1$

$\sin^2\theta = 1$

$\theta = \pi/2$

$$\beta(m,n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1}(1-\sin^2\theta)^{n-1}\, 2\sin\theta\cos\theta\,d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-2}\theta \cdot (\cos^2\theta)^{n-1} \sin\theta\cos\theta\,d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-2}\theta \cdot \cos\theta\,d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta\,d\theta$$

4. $\beta(m,n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$

Proof: Put $y = \frac{x}{1+x}$

$y + xy = x$

$y = x - xy$

$x = \frac{y}{1-y}$

$$dx = \frac{(1-y)dy - y(0 \cdot dy)}{(1-y)^2}$$

$$= \frac{dy - ydy + ydy}{(1-y)^2} \rightarrow dx = \frac{dy}{(1-y)^2}$$

when $x = 0$, $y = \frac{0}{1+0} = \frac{0}{1} = 0$

when $x = \infty$, $y = \frac{x}{1+x} = \frac{x}{x(1/x + 1)}$

$$= \frac{1}{1/x + 1}$$

$$= \frac{1}{1/\infty + 1} = \frac{1}{0+1} = 1$$

$$\int_0^\infty \frac{x^{m-1}}{(1+x)^{n+m}} dx = \int_0^1 \frac{\left(\frac{y}{1-y}\right)^{m-1} \frac{dy}{(1-y)^2}}{\left(1+\frac{y}{1-y}\right)^{m+n}}$$

$$= \int_0^1 \frac{\frac{y^{m-1}}{(1-y)^{m+1}} dy}{\left(\frac{1-y+y}{1-y}\right)^{m+n}}$$

ie

$$= \int_0^1 \frac{y^{m-1}}{(1-y)^{m+1}} (1-y)^{m+n} dy$$

$$= \int_0^1 y^{m-1} \cdot (1-y)^{m+n-m-1} dy$$

$$= \int_0^1 y^{m-1} \cdot (1-y)^{n-1} dy$$

$$= \beta(m, n)$$

• Show that $\int_0^1 \frac{x^{m-1} + n^{n-1}}{(1+x)^{m+n}} \cdot dy =$

Proof:

$$\int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}} dx + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}} dx$$

Take $\int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}} dx$

Put $x = 1/y$

$dx = \frac{-1}{y^2} dy$

$y = 1/x$

when $x=0,\ y=1/0=\infty$

when $x=1,\ y=1$

$$\int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\,dx = \int_\infty^1 \frac{(1/y)^{n-1}}{(1+1/y)^{m+n}}\left(-1/y^2\right)dy$$

$$= -\int_\infty^1 \frac{\frac{1}{y^{n-1}}}{\left(\frac{y+1}{y}\right)^{m+n}} \cdot 1/y^2 \cdot dy$$

$$= \int_1^\infty \frac{\frac{1}{y^{n+1}}}{\frac{(1+y)^{m+n}}{y^{m+n}}}\,dy$$

$$= \int_1^\infty \frac{y^{m+n-(n+1)}}{(1+y)^{m+n}}$$

$$= \int_1^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\,dy$$

$$= \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

$$\int_0^\infty \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\,dx + \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

$$= \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

$$= \beta(m,n)$$

→ Prove that $\displaystyle\int_0^\infty \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx = 2\beta(m,n)$

$$\int_0^\infty \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}} = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx + \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

$$= \beta(m,n) + \beta(n,m)$$

$$= \beta(m,n) + \beta(m,n)$$

$$= 2\beta(m,n)$$

→ Show that $\displaystyle\int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}}\,dx = \frac{1}{a^n b^m}\beta(m,n)$

$$\int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}} = \int_0^\infty \frac{x^{m-1}}{a^{m+n}(1+bx/a)^{m+n}}\,dx$$

Put $\frac{b}{a}x = z$

$x = \frac{a}{b}z \qquad dx = \frac{a}{b}\,dz$

when $x=0$, $z=0$

when $x=\infty$ $z=\infty$

$$\int_0^\infty \frac{x^{n-1}\,dx}{(a+bx)^{m+n}} = \frac{1}{a^{m+n}}\int_0^\infty \frac{(a/b\, z)^{m-1}}{(1+z)^{m+n}} \cdot a/b\, dz$$

$$= \frac{1}{a^{m+n}} \cdot \frac{a^{m-1}}{a^{n-1}} \cdot \frac{a}{b} \int_0^\infty \frac{z^{m-1}\,dz}{(1+z)^{m+n}}\, dz$$

$$= \frac{1}{a^n b^m}\, \beta(m,n)$$

• Express integral $\int_a^b (x-a)^{m-1}(b-x)^{n-1}\,dx$ in terms of $\beta$ function

Put $x = a+(b-a)z$

$dx = (b-a)\,dz$

when $x=a$

$a = a+(b-a)z$

$0 = (b-a)z$

$z = 0$

when $x=b$

$b = a+(b-a)z$

$b-a = (b-a)z$

$z = 1$

$$\int_a^b (x-a)^{m-1}(b-x)^{n-1}\,dx$$

$$= \int_0^1 \left((b-a)z\right)^{m-1} \left(b-(a+(b-a)z)\right)^{n-1} (b-a)\,dz$$

$$= \int_0^1 (b-a)^{m-1} z^{m-1} \cdot \left\{(b-a)-(b-a)z\right\}^{n-1} (b-a)\,dz$$

$$= (b-a)^{m-1} \cdot (b-a)^{n-1} \cdot (b-a) \int_0^1 z^{m-1} (1-z)^{n-1}\,dz$$

$$= (b-a)^{m+n-1}\, \beta(m,n)$$

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$$\Rightarrow \int_0^a (a-x)^{m-1} x^{n-1}\,dx$$

$$= \int_0^a a^{m-1}\left(1-\frac{x}{a}\right)^{m-1} x^{n-1}\,dx$$

put $x/a = z \Rightarrow x = az$ $\quad dx = a\,dz$

when $x = 0$, $z = 0$

when $x = a$, $z = 1$

$$\int_0^a (a-x)^{m-1} x^{n-1} \, dx = a^{m-1} \int_0^1 (1-z)^{m-1} (az)^{n-1} a \, dz$$

$$= a^{m-1} \cdot a^{n-1} \cdot a \int_0^1 z^{n-1} (1-z)^{m-1} \, dz$$

$$= a^{m+n-1} \beta(m,n)$$